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GRE math study group
Linear algebra examples
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Linear algebra is one of the topics covered by the GRE test in mathematics. Here are the questions relating to linear algebra on the sample test with a bit of discussion for each one

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3. If V and W are 2-dimensional subspaces of $\mathbf{R}^4$, what are the possible dimensions of the subspace $V \cap W$?

(A) 1 only   (B) 2 only   (C) 0 and 1 only   (D) 0, 1, and 2 only   (E) 0, 1, 2, 3, and 4

A 2-dimensional subspace in 4-space is just a plane in 4-space that passes through the origin. How can two planes intersect?

In 3-space, your experience gives the answer. If they're not the same plane, then they must intersect in a line. (They have the origin in common, so they can't be parallel.) V could be the same plane as W, and in that case, their intersection is that plane. So, in $\mathbf{R}^3$, the possible dimensions are 1 and 2.

But 4-space is beyond our experience. We can still get 1 and 2 dimensional intersections, but can we get other dimensions. You certainly can't get 3, because the intersection can't be bigger than V or W. That's enough to answer the question. It's got to be (D).

In fact, two planes through the origin in 4-space can intersect only at the origin. For example, let one plane be the x, y-plane where $z = w = 0$, and let the other one be the z, w-plane where $x = y = 0$. They intersect only at the origin.

Is there a general theorem from linear algebra which says what the possible dimensions are? Yes. Besides the intersection of two subspaces V and W, there's also a span of them, sometimes called their sum $V + W$. It's the smallest subspace that contains both V and W. There's a theorem which says $\dim(V) + \dim(V) = \dim(V \cap W) + \dim(V + W)$. That allows you to determine what possible dimensions $V \cap W$ can have.

For example, in $\mathbf{R}^{10}$, if $\dim(V) = 6$ and $\dim(W) = 7$, then $\dim(V \cap W) + \dim(V + W) = 13$. But $\dim(V + W) = 13$ has to be at least 7 and can't be greater than 10, so $\dim(V \cap W)$ can range from 3 through 6.

12. Let A be a 2×2 matrix for which there is a constant k such that the sum of the entries in each row and each column is k. Which of the following must be an eigenvector of A?

I. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
II. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
III. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(A) I only   (B) II only   (C) III only   (D) I and II only   (E) I, II, and III

An eigenvector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is one that A maps to a multiple of itself, that is, there's some number λ so that $A\mathbf{x} = \lambda\mathbf{x}$. Let our matrix be $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the matrix equation $A\mathbf{x} = \lambda\mathbf{x}$ says

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

One reason that eigenvectors are important is that you can use them to understand what the transformation is doing geometrically.

If the rows and columns in A each add to the same number k, then $a + b = a + c = b + d$, so $b = c$, and $a = d$. So our matrix looks like $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, and then our equation looks like

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

which we can write as two equations

$$ax + by = \lambda x$$
$$bx + ay = \lambda y$$

Now, are any of the three named vectors eigenvectors? Just the vector III. That works because the two equations each say $a + b = \lambda$ which tells you what the constant λ is. So the answer is (C).

18. Let V be the real vector space of all real 2×3 matrices, and let W be the real vector space of all real 4×1 column vectors. If T is a linear transformation from V <u>onto</u> W, what is the dimension of the subspace $\{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$?

(A) 2   (B) 3   (C) 4   (D) 5   (E) 6

V is a 6-dimensional space while W is a 4-dimensional space. Since T squeezes 6 dimensions into 4, 2 dimensions have to be squeezed to 0, so the answer is A.

The subspace sent to $\mathbf{0}$ is called the *null space* or *kernel* of the transformation. There's a general theorem that says $\dim(V) = \dim(\text{kernel}) + \dim(\text{image})$. In this problem, the dimension of the image is 4 since T maps onto W. The answer is (C).

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27. Consider the two planes $x + 3y - 2z = 7$ and $2x + y - 3z = 0$ in $\mathbf{R}^3$. Which of the following sets is the intersection of these planes?

(A) $\emptyset$
(B) $\{(0, 3, 1)\}$
(C) $\{(x, y, z) : x = t, y = 3t, z = 7 - 2t, t \in \mathbf{R}\}$
(D) $\{(x, y, z) : x = 7t, y = 3 + t, z = 1 + 5t, t \in \mathbf{R}\}$
(E) $\{(x, y, z) : x - 2y - z = -7\}$

In 3-space, the intersection of two planes is empty if and only if they're parallel. But to be parallel, the left sides of the equations have to be the same or constant multiples of each other. For example, $x + 3y - 2z = 8$ is parallel to the first plane, and $2x + 6y - 4z = 16$ also describes that plane. But the two given planes aren't parallel, so the answer can't be (A).

Two planes in 3-space can't intersect in a single point, so the answer can't be (B) either even though the point in $(0, 3, 1)$ is in the intersection.

(C) describes a line, which is what we're looking for, but it can't be right because it doesn't include the point $(0, 3, 1)$.

(E) can't be right either, because it describes a plane.

That leaves (D). You could also solve the two equations simultaneously to find the solution, but when you do, you won't parameterize the solution exactly as shown in (D), so you'll have to manipulate it to get it in that form.

36. Let M be a 5×5 real matrix. Exactly four of the following five conditions on M are equivalent to each other. Which of the five condtions is equivalent to NONE of the other four?

(A) For any two distinct column vectors $\mathbf{u}$ and $\mathbf{v}$ of M, the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

(B) The homogeneous system $M\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(C) The system of equations $M\mathbf{x} = \mathbf{b}$ has a unique solution for each real 5×1 column vector $\mathbf{b}$.

(D) The determinant of M is nonzero.

(E) There exists a 5×5 real matrix N such that NM is the 5×5 identity matrix.

Four of these are logically equivalent and you'll recognize two or three of them from linear algebra. They describe when a square matrix is invertible, also called nonsingular, and what that means when the matrix is the coefficient matrix for a system of linear equations.

The last condition (E) is just the definition of invertible. The matrix N is called the inverse of M and denoted M^{-1}.

You can use it to solve the systems of linear equations in (B) and (C). Multiply the equation $M\mathbf{x} = \mathbf{0}$ on the left by M^{-1} to get $M^{-1}M\mathbf{x} = M^{-1}\mathbf{0}$ and simplify to get $\mathbf{x} = \mathbf{0}$. Thus, (B) only has the trivial solution. Likewise, the only solution to (C) is $\mathbf{x} = M^{-1}\mathbf{b}$.

Since the determinant of the product equals the product of the determinant, if we take the determinant of both sides of the equation $M^{-1}M = I$, we get $|M^{-1}||M| = 1$, and that implies that the determinant $|M|$ is not zero.

Furthermore, each of (B), (C), and (D) imply (E).

But (A) is a weaker statement. It says that the rank of the matrix is at least 2, and only rank 5 matrices are invertible. If the five columns of the matrix are $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}+2\mathbf{v}$, and $\mathbf{u}+3\mathbf{v}$, then any two of them are independent, but any three are not independent.

50. Let A be a real 2×2 matrix. Which of the following statements must be true?

I. All of the entries of A^2 are nonnegative.

II. The determinant of A^2 is nonnegative.

III. If A has two distinct eigenvalues, then A^2 has two distinct eigenvalues.

(A) I only   (B) II only   (C) III only   (D) II and III only   (E) I, II, and III

These conditions are unrelated, but in questions like this, you can often narrow down the answer with partial knowledge and choose among the remaining possibilities.

Condition II is probably the easiest one to answer. $|A^2| = |A|^2$, so the determinant of A^2 is a square, and, therefore, it can't be negative. So II is true. That rules out (A) and (C).

If you know about eigenvalues, you can determine if III is true. If A has an eigenvalue λ, then $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$. Therefore $A^2\mathbf{x} = \lambda^2\mathbf{x}$. Thus, the eigenvalues for A^2 are the squares of the eigenvalues of A. That means III is false, because A could have eigenvalues 2 and -2, but then A^2 would have only the eigenvalue 4. Since III is false, that rules out (C), (D), and (E).

By now, we know the answer is (B).

To directly determine if I is true, you could look at a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, square it, and see if there's some reason the entries can't be negative. There is no reason, and you can construct an example once you see the entries in A^2. For instance, the upper left entry is $a^2 - bc$, so set $a = 0$, $b = c = 1$, and d to anything, and you'll have a counterexample.

58. Suppose A and B are $n \times n$ invertible matrices, where $n > 1$ and I is the $n \times n$ identity matrix. If A and B are similar matrices, which of the following statements must be true?

I. $A - 2I$ and $B - 2I$ are similar matrices.

II. A and B have the same trace.

III. A^{-1} and B^{-1} are similar matrices.

(A) I only   (B) II only   (C) III only   (D) I and III only   (E) I, II, and III

Matrices are said to be similar if they describe the same linear transformation but use different bases. Algebraically, that means there is some invertible matrix P, called the *change of basis matrix*, such that $P^{-1}AP = B$.

Since similar matrices describe the same linear transformation, they have a lot of the same properties. An invariant of the linear transformation is anything you can tell about the transformation just by looking at the geometry of the transformation. That includes eigenvalues, the determinant (which is the product of the eigenvalues), and the trace (which is the sum of the eigenvalues).

Since they have the same trace, II is true.

In fact, so are I and III.

Statement I. Since $P^{-1}AP = B$, therefore $P^{-1}(A - 2I)P = P^{-1}AP - 2P^{-1}IP = B - 2I$.

Statement II. Invert both sides of the equation $P^{-1}AP = B$ to get $(P^{-1}AP)^{-1} = B^{-1}$. That simplifies to $P^{-1}A^{-1}P = B^{-1}$.

Thus (E) is the correct answer.

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