Math 105 History of Mathematics
First Test
March 2017

Scale. 90–100 A. 75–89 B. 60–74 C. Median 90.

Problem 1. [30] Essay.

Select *one* of the three topics A, B, and C. Please think about these topics and make an outline before you begin writing. You will be graded on how well you present your ideas as well as your ideas themselves. Each essay should be relatively short, one to three written pages. There should be no fluff in your essays. Make your essays well-structured and your points as clearly as you can.

Topic A. We've discussed mathematics of Egypt, Babylonia, and Greece. Briefly summarize their transmission between cultures. Identify mathematics that may have been transmitted from one culture to one of the others. Explain why you think it may have been transmitted.

There seems to be no evidence of transmission between Egyptians and Babylonians; their number systems and computational methods were different and some of their knowledge of geometry was different. On the other hand, the Greeks learned much from both of the earlier cultures. For instance, Greeks used both unit fractions from the Egyptians and sexagesimal numbers from the Babylonians. There are several stories about early Greek mathematicians visiting both Egypt and Babylonia and learning mathematics there.

Topic B. Famous geometry problems. Select one of the following three famous problems and write about it. State the problem clearly and completely. Explain why the problem was considered important by Greek mathematicians. Describe one or two attempts to solve the problem by Greek mathematicians. Explain why the attempts you describe were not considered satisfactory. Explain how, if at all, those attempts led to further investigations in mathematics.

B(i). Duplication of the cube (the Delian problem).
B(ii). Angle trisection.
B(iii). Quadrature of the circle

Here are some things you might say in each of the essays. Not everything listed needs be said in an essay, and you may have thought of other important points.

i. Duplication of the cube. This problem, also known as the Delian problem, was to construct a cube of twice the volume of a given cube. (The story goes that to staunch the plague at Athens, the oracle of Apollo at Delos required that the cubic altar be doubled.) Ideally, the construction should be made only using the compass and straightedge. Of course, we would say the problem is to construct a line segment the cube root of two times as long as a given segment. The analogous problem of doubling a square is easy—just make a square on the diagonal of the given square. It wasn't until modern times that it was proved that the Delian problem is unsolvable with straightedge and compass alone.

Hippocrates of Chios reduced the problem to finding two mean proportionals, that is, given a and b, find x and y so that $a : x = x : y = y : b$, for when $a = 1$ and $b = 2$, then x is the cube root of two. This, of course, was not a solution since there was no known way at that time of finding two mean proportionals, and, furthermore the theory of ratios would not be satisfactorily developed until Eudoxus. Still, it was an insight into the problem that was helpful to later mathematicians.

Archytas found a three-dimensional solution depending on intersecting a cone, a cylinder, and a torus, but their constructions go beyond the tools of straightedge and compass.

Menaechmus found other solutions depending on intersecting parabolas and hyperbolas, curves which he invented for the purpose, whose construction also depended on more than straightedge and compass. These conic sections became favorite curves of study ever since. Also, Menaechumus' suggestion of coordinates might have been important if it had been developed, but unfortunately it was not developed until Fermat and Descartes in the 17th century.

Archimedes had another solution of the Delian problem which also depended on intersecting a parabola and a hyperbola. Indeed, Archimedes used conic sections to solve the general cubic equation, not just for finding cube roots.

One can say that the quest to double the cube encouraged the study of ratios and proportions and of solids of revolution, led to the invention of conic sections, could have led to coordinate geometry, and suggested solutions to the general cubic equation.

ii. Angle trisection. The problem of angle trisection was to divide any given angle into three equal parts by means of the classical tools of geometry, the straightedge and compass. Angle bisection is easy, and division of a line segment into any number of equal parts is easy, but it wasn't until recent times that it was shown that it is impossible to trisect any given angle with straightedge and compass alone. It is likely that one of the sources of this problem was the goal to construct regular polygons, in particular, those where the number of sides is divisible by nine.

Hippias described a curve he called a trisectrix, which he used to trisect angles. It is the locus of the intersection of a uniformly moving line and a uniformly rotating ray. It allows the interconversion of angle measurement and distance, and, therefore, may be used to divide an angle into any number of equal parts (since a line segment can be divided into any number of equal parts). It was not considered a satisfactory solution since the generation of this curve depends on more than a straightedge and compass.

Archimedes spiral is also generated by a moving point, this time, one which moves uniformly alone a ray while the ray rotates uniformly. Since it can also be used to convert between angle measurement and distance, it can also be used to solve the trisection problem. It, of course, suffered from the same fault at Hippias' solution.

Other solutions due to Archimedes, Apolonius, and Pappus used only the straightedge and compass. Unfortunately, the straightedge was not used to only to draw lines between pairs of points, but was fitted between two given curves so that it passed through a given point. Hence, this solution was not justified by Euclid's axioms of geometry and was not considered satisfactory.

The quest for trisecting an angle lead to the invention of curves based on moving lines. If motion had been analyzed to the same degree as static geometry had been, the Greeks might have developed differential calculus, but it was not considered a proper field of study by most mathematicians of the time.

iii. Quadrature of the circle. This problem was to find a square equal to a given circle, that is, given the diameter of a circle, find the side of a square whose area is the same as the area of the circle. We would say the side of the square is the square root of pi times half of the diameter, and so the solution is related to the value of π. The Egyptians had an approximation to the answer, namely, take the side of the square to be $\frac{8}{9}$ of the diameter. The Babylonians also had approximations of the area of a circle. Approximations are nice, but the problem is to construct exactly the side of a square, and to do it only using the classical tools of geometry, the straightedge and compass. Only in recent times has it been shown that those tools are insufficient for the construction.

Hippocrates knew that circles are in the same ratios as squares on their diameters, and that similar segments of circles are in the same ratios as the squares on their bases. Although he used these statements, it is doubtful that he could prove them as the theory of ratios and proportions would not be established and the method of exhaustion would not be developed until later by Eudoxus. Using these statements, Hippocrates could find quadratures of certain lunes by straightedge and compass alone. This gave hope that circles themselves could be squared.

Dinostratus showed that Hippias' trisectrix curve could be used for circle squaring as well as trisecting angles, but as the trisectrix depends on more than straightedge and compass for its construction, Dinostratus' solution was not considered satisfactory.

After Eudoxus finally proved that circles are in the same ratios as squares on their diameters using what we call the method of exhaustion, the area of circles could be approximated arbitrarily closely, but not exactly. Archimedes later carried out this approximation to give an approximation of $\frac{22}{7}$ for π and closer approximations as well. These may be practically useful, but, of course, could not be considered to be solutions to the problem.

Archimedes also showed that his spiral could be used to square the circle, but as it depended on motion for its generation (as did Hippias' trisectrix), this was not considered to be a proper solution.

The quest for squaring the circle lead to the actual quadrature of some curved figures (Hippias' lunes), spurred the development of the theory of ratios and proportions, lead to the method of exhaustion, and brought us closer approximations to π than had been known before.

Topic C. Babylonian arithmetic. Describe the numerals that Babylonians used, how they represented fractions, and their algorithms for addition, subtraction, multiplication, and division. Explain at least one method they used for finding square roots.

Summary of points to make in your essay. Numerals were written with one symbol for 1 and another for 10. A place value system in base sixty was used so that six of the symbols for 10 was equal to one of the symbols for 1 in the next column to the left. Blanks were used for 0. For example, the number 500, which is 8 times 60 plus 20, in base 60 becomes 8,20 (actually eight 1-symbols followed by two 10-symbols).

Fractions were also written in base 60 so that 3/4 becomes 0;45 in base 60, but decimal points and zeros didn't appear, so 3/4 would look like 45. (The semicolons and commas we use to transcribe their numbers don't correspond to anything in their writing. We use them to help us understand what they had to tell by context.)

Since it's a place-value system, addition, subtraction, and multiplication algorithms are the same ones we use in base 10, except done base 60, and so they're a bit more complicated. They didn't use long division, however. Instead they used tables to look up the reciprocal of the divisor and multiplied that by the dividend.

They used several methods for finding square roots. The simplest was just a table of squares and square roots, and if what they were looking for wasn't in the table, then they used linear interpolation. They also used a couple of other algorithms described in the text.

Problem 2. [10] Find the greatest common divisor of the two numbers 1834 and 1274 by using the Euclidean algorithm. (Computations are sufficient, but show your work. An explanation is not necessary.)

Repeatedly subtract the smaller number from the larger until the smaller divides the larger without remainder. That smaller is the GCD. You can speed things up a bit by replacing the larger by the remainder when divided by the smaller. For these two numbers, the GCD is 14.

Problem 3. [20] Short essay on Euclid's *Elements*.

The *Elements* is the earliest extant example of formal mathematics. Describe its structure and how formal mathematics depends on such a structure. (One or two paragraphs should be sufficient.)

Formal mathematics requires careful definitions and clear proofs.

The *Elements* begins with definitions and axioms (both common notions and postulates). Some of these just describe the terms to be used, others are more substantive and state specific assumptions about properties of the mathematical objects under study. Definitions are also given for new concepts stated in terms of the old concepts as the new concepts are needed. Propositions are stated one at a time only using those terms already introduced, and each proposition is proved rigorously. The proof begins with a detailed statement of what is given and what is to be proved. Each statement in the proof can be justified by definitions, axioms, previously proved propositions, or as an assumption in the beginning of a proof by contradiction. The last statement in a proof is that which was to be proven.

This structure (where each proof depends on previously proved propositions, definitions, and axioms) is required to prevent circular arguments. The definitions and axioms are starting points for the theory.

Problem 4. [20; 10 points each part] On Egyptian arithmetic.

a. Illustrate how the Egyptian multiplication algorithm works by computing 45 times 97 (which is 4365).

You either can start with a line with 1 and 45 in it and repeatedly double that line until you find a sum of powers of 2 in the left column that give 97, or you can start with a line with 1 and 97 and repeatedly double that line until you find a sum of powers of 2 giving 45. In either case, add the corresponding entries in the second column to find the product.

b. Illustrate how Egyptian division algorithm works by computing 4365 divided by 97 (which is 45).

Start with the line of 1 and 97 and repeatedly double it until you find a sum of numbers in the right column that add to 4365, then add the corresponding entries in the left column to find the quotient 45.

Problem 5. [20; 4 points each part] True/false. For each sentence write the whole word "true" or the whole word "false". If it's not clear whether it should be considered true or false, you may explain in a sentence if you prefer.

a. A common ancient approximimation for the circumference of a circle was three times its diameter. *True.*

b. Whereas the Egyptians wrote on clay tablets, the Babylonians used papyrus. *False.* That's just backwards.

c. Negative numbers were accepted in Egypt but not in Babylonia or Greece. *False.* None of them used negative numbers.

d. Perfect numbers are whole numbers whose only prime factors are 2, 3, and 5. *False.* A number if perfect if it is the sum of its proper divisors, like $28 = 1 + 2 + 4 + 7 + 14$.

e. Euclid gave constructions of regular n-gons for $n = 5, 6, 7, 8$, and 9. *False.* No, a 9-gon can't be constructed with Euclidean tools since it requires trisection of an angle. Also, 7-gons can't be constructed. He did give constructions for the others.