Math 114 Discrete Math

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Scale: 89–100 A, 73–88 B, 59–72 C.

Problem 1. Translation into symbolic expressions. [20; 5 points each part] Let P, Q, and R be abbreviations for the following predicates.

$P(x)$: x is perfect
$Q(x)$: x failed the quiz
$R(x)$: x read a lot of books

Write these propositions using P, Q, R, logical connectives, and quantifiers.

a. Everyone who is perfect read a lot of books.

$\forall x\,(Px \to Rx)$.

Note: with universal quantifiers, remember to put assumptions in the premise of an if/then statement. The expression $\forall x\,(Px \wedge Rx)$ means everyone is perfect and read a lot of books.

b. No one who read a lot of books failed the quiz.

$\neg\exists x\,(Rx \wedge Qx)$.

Note: with existential quantifiers, remember to put assumptions in conjuncts. The expression $\neg\exists x\,(Rx \to Qx)$ is logically equivalent to $\forall x\,(Rx \wedge \neg Qx)$ which says everyone read a lot of books and passed the quiz.

c. Someone who failed the quiz isn't perfect.

$\exists x\,(Qx \wedge \neg Px)$.

d. If everyone reads a lot of books, then no one will fail the quiz.

$(\forall x\,Rx) \to (\neg\exists x\,Qx)$.

Note: Two quantifiers are needed. The expression $\forall x\,(Rx \to \neg Qx)$ says whoever read a lot of books will pass the quiz, and that's a stronger statement than the one given.

Problem 2. On truth tables. [20; 10 point each part]

a. Use a truth table to show that $(p \to q) \wedge (p \to r)$ is logically equivalent to $p \to (q \wedge r)$. Explain in a sentence why your truth table shows that they are logically equivalent.

p	q	r	$(p \to q)$	$\wedge$	$(p \to r)$	$p \to$	$(q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	F
F	F	T	T	T	T	T	F
F	F	F	T	T	T	T	F

Since the two expressions have the same truth value in all eight cases, $TFFFTTTT$, therefore they are logically equivalent.

b. Use a truth table to show that $(p \wedge q) \to r$ is not logically equivalent to $(p \to r) \wedge (q \to r)$. Explain in a sentence why your truth table shows that they aren't logically equivalent.

p	q	r	$(p \wedge q)$	$\to r)$	$(p \to r)$	$\wedge$	$(q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the two expressions don't have the same truth value in all eight cases (they differ in the 4th and 6th cases), they are not logically equivalent.

Problem 3. Interpretation of symbolic expressions. [20; 5 points each part] Determine the truth value of each of the following statements if the universe of discourse for each variable consists of all real numbers. Simply write "true" or "false" for each; no need to explain why.

a. $\forall x\,\exists y\,(y = 3x + 2)$.
True. Given an x, triple it and add 2 to get y.

b. $\exists y\,\forall x\,(y = 3x + 2)$.
False. When you triple x and add 2, you don't always get the same number y.

c. $\exists x\, \exists y\, (x^2 + y^2 = 1)$.

True. Any point on the unit circle gives a solution to the equation.

d. $\forall y\, \forall x\, (x + y)^2 = x^2 + y^2$.

False. One counterexample is enough. For instance, $(2 + 3)^2 \neq 2^2 + 3^2$.

e. $\forall x\, (x < 0 \vee x = 0 \vee x > 0)$.

True. This is called the law of trichotomy.

Problem 4. On rules of inference. [20] Given statements 1 and 2, find an argument for the conclusion C. For each intermediate statement you make, state what rule of inference you use and the number(s) of the previous lines that rule uses.

(You don't have to use symbolic notation for this problem, but it may help. Also, if you don't remember the name for a rule of inference, then just state the whole rule symbolically.)

1. Linda, a student in this class, owns a red convertible.

2. Everyone who owns a red convertible has gotten at least one speeding ticket.

C. Someone in this class has gotten a speeding ticket.

I'll give an answer that uses symbols.

$S(x)$: x is a student in this class
$R(x)$: x owns a red convertible
$T(x)$: x has gotten a speeding ticket.

1.	$S(\text{Linda}) \wedge R(\text{Linda})$	Given
2.	$\forall x\, (Rx \to Tx)$	Given
3.	$R(\text{Linda}) \to T(\text{Linda})$	$2, \forall$-elimination
4.	$R(\text{Linda})$	1, simplification
5.	$T(\text{Linda})$	3, 4, Modus ponens
6.	$S(\text{Linda})$	1, simplification
7.	$S(\text{Linda} \wedge T(\text{Linda})$	5, 6, conjunction
8.	$\exists x\, (Sx \wedge Tx$	$7, \exists$-introduction

Thus, we have proven C as statement 8.

Problem 5. On proofs. [20] Recall that an integer n is *even* iff $\exists k, n = 2k$. Also, an integer n is *odd* iff $\exists k, n = 2k + 1$. Furthermore, each integer is either even or odd, but no integer is both even and odd.

a. [15] Show that if n^2 is an odd integer, then n is also an odd integer. (Think about this before writing down your proof; you may even want to work it out on the back of another page, then write your final proof clearly here. You don't have to name any rules of inference like you did in the previous problem.)

There are many proofs you could use. Here's one that shows the contrapositive: if n is not an odd integer, then n^2 is not an odd integer.

Suppose that n is not odd. Then n even, and so it has the form $n = 2k$. Therefore, $n^2 = 4k^2 = 2(2k^2)$, so n^2 is also even. Hence, n^2 is not odd. Q.E.D.

b. [5] Was your proof in part **a** a direct proof, an indirect proof (by contraposition), or a proof by contradiction?

The proof above was by contraposition, but your proof might have a different form.