Calculus II – Math 121

Test #2

Spring 2007

For the last 10 points on this test, please hand in correction problem for each problem on the test that you lost any points.

Problem 1 – Section 5.7 #18
Problem 2 – Section 5.7 #60
Problem 3 – Section 5.7 #50
Problem 4 – Section 5.9 #22
Problem 5 – Section 6.2 #24
Problem 6 – Section 6.3 #22

1. [10 points each] Compute the following indefinite integrals:

a. $\displaystyle\int \left(x^7 + (1+x)^7 + (1-x)^7\right)\, dx$

You could do this by inspection. That is, you could guess that the answer should look something like $\frac{1}{8}x^8 + \frac{1}{8}(1+x)^8 + \frac{1}{8}(1-x)^8 + C$, check by taking the derivative and see that the third term is off by a factor of -1 and negate it to fix it.

Alternatively, you could break it into three integrals.

$$\int x^7\, dx + \int (1+x)^7\, dx + \int (1-x)^7\, dx$$

The first integral is clear, the second you could do with a substitution $u = 1 + x$, and the third with a substitution $u = 1 - x$.

Either way, you should get the answer

$$\frac{1}{8}x^8 + \frac{1}{8}(1+x)^8 - \frac{1}{8}(1-x)^8 + C.$$

b. $\displaystyle\int 5x\sqrt{1-3x^2}\, dx$

Again, you could find the integral by inspection, but here's a formal way using substitution. Since there is a composition where the inner function is $1 - 3x^2$, you should try the substitution $u = 1-3x^2$. For that substitution, $du = -6x\, dx$, so the integral becomes $\int -\frac{5}{6}\sqrt{u}\, du$ which is clearly $-\frac{5}{9}u^{3/2}+C$, which, upon substituting back gives the answer

$$-\frac{5}{9}\,(1-3x^2)^{3/2} + C.$$

c. $\displaystyle\int 2\sin 3x\, \cos 3x\, dx$

You might do this with two substitutions, the first being $u = 3x$, $du = 3\, dx$. That simplifies the integral a little bit to

$$\int \tfrac{2}{3}\sin u\, \cos u\, du.$$

At this point you might see note that the derivative of sine is cosine, suggesting the substitution $v = \sin u$, $dv = \cos u\, du$, which gives the integral

$$\int \tfrac{2}{3} v\, dv.$$

Alternatively, you might do just one substitution $v = \sin 3x$, $dv = 3\cos 3x\, dx$, which does the same thing as the two substitutions mentioned above. In any case, that last integral equals $\frac{1}{3}v^2 + C = \frac{1}{3}\sin^2 u + C = \frac{1}{3}\sin^2 3x + C$.

2. [10 points each] Compute the following definite integrals:

a. $\displaystyle\int_{-1}^{1} \frac{t}{(1+t^2)^3}\, dt$

The easiest way to find the integral is to note that the integrand is an odd function and it's evaluated from $-a$ to a, so its integral is 0. But if you didn't notice that, there are other ways to evaluate it.

Since there's a composition where the inner function is $1+t^2$, the substitution $u = 1+t^2$, $du = 2t\, dt$ ought to help. Note that when $t = -1$, $u = 2$, and when $t = 1$, $u = 2$ also, so the integral becomes

$$\int_2^2 \frac{1}{2u^3}\, du.$$

Now since the limits of integration are the same, therefore the value of the integral is 0.

b. $\displaystyle\int_1^4 \frac{(1+\sqrt{x})^3}{\sqrt{x}}\, dx$

Either of the substitutions $u = \sqrt{x}$ or $u = 1 + \sqrt{x}$ work, but the second one happens to work a little better, and it probably is the one you chose since there's a composition where the inner function is $1 + \sqrt{x}$.

If $u = 1 + \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}\, dx$, then when $x = 1$, $u = 2$, and when $x = 4$, $u = 3$, so the integral becomes

$$\int_2^3 2u^3\, du = \tfrac{1}{2}u^4\Big|_2^3 = \tfrac{1}{2}(3^4 - 2^4) = \tfrac{65}{2}.$$

3. [10] Use the substitution $u = 1 + x^3$ to transform the integral $\displaystyle\int_1^2 \frac{x^2\sin(1+x^3)}{\sin(1-x^3)}\, dx$ into an integral with respect to u. (Do not evaluate the integral.)

If $u = 1 + x^3$, then $du = 3x^2\, dx$. For x, the limits of integration are 1 and 2, so for u the limits of integration are $1 + 1^3 = 2$ and $1 + 2^3 = 9$. The factor $\sin(1+x^3)$ in the numerator will become $\sin u$. The factor $x^2\, dx$ will become $\frac{1}{3}\, du$. That leaves the denominator $\sin(1-x^3)$ left to be converted. But $u = 1+x^3$, so $x^3 = u-1$, and $1-x^3 = 2-u$, so $\sin(1-x^3) = \sin(2-u)$. Thus, the original integral is converted to

$$\int_2^9 \frac{\sin u}{3\sin(2-u)}\, du.$$

4. [10] **a.** What is the average value of $y = \sin x$ on the interval $[0, \pi]$?

The average value of a function f on an interval $[a, b]$ is the integral of f divided by the length of the interval. Thus

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{\pi}\int_0^\pi \sin x\,dx \\ &= \frac{1}{\pi}(-\cos x)\Big|_0^\pi \\ &= \frac{-\cos\pi + \cos 0}{\pi} = \frac{2}{\pi} \end{aligned}$$

b. (Circle the correct conclusion.) The *Mean Value Theorem for Integrals* implies that there is a value c, $0 < c < \pi$, such that

i. $\sin c = \dfrac{0+\pi}{2}$.

ii. $\sin c = \dfrac{\sin 0 + \sin\pi}{2}$.

iii. $\sin c = \dfrac{\cos 0 + \cos\pi}{2}$.

iv. $\sin c = \dfrac{\cos 0 - \cos\pi}{\pi}$.

v. none of these.

The MVT says that a continuous function takes on its average value. Sine is a continuous function, and in part a you showed that its average value was equal to the expression shown in iii.

5. [10] Consider the plane region bounded by the curves $y = 1$, $x = 1$ and $y = 2 - (x-1)^2$. Rotate this region about the x-axis to generate a solid of revolution.

Use the disk or washer method, whichever is appropriate, to write an integral which gives the volume of that solid of revolution. (Do not evaluate the integral.)

A vertical cross section of the plane region at x, when rotated around the x-axis, generates an annulus (washer) with outer radius $f(x) = 2-(x-1)^2$ and inner radius $g(x) = 1$. The area of this annulus is

$$\pi f(x)^2 - \pi g(x)^2 = \pi(2-(x-1)^2)^2 - \pi 1^2$$

and, so, the integral giving the volume of the solid is

$$\int_0^1 \left(\pi(2-(x-1)^2)^2 - \pi\right) dx.$$

Note that the limits of integration are 0 and 1 since those are the left and right sides of the plane region.

6. [10] Consider the same plane region bounded by the curves $y = 1$, $x = 1$ and $y = 2 - (x-1)^2$. Rotate this region about the y-axis to generate a solid of revolution.

Use the shell method to write an integral which gives the volume of that solid of revolution. (Do not evaluate the integral.)

A vertical cross section of the plane region at x, when rotated around the y-axis, generates a cylinder (shell) with radius x, circumference $2\pi x$, and height $f(x) - g(x) = 2 - (x-1)^2 - 1 = 1 - (x-1)^2$. Therefore the area of this shell is

$$2\pi x(1-(x-1)^2).$$

so the volume of the solid is

$$\int_0^1 2\pi x(1-(x-1)^2)\,dx.$$