Basic Rules of Integration

D Joyce

Clark University

The basic rules of integration correspond to those of differentiation through the Fundamental Theorem of Calculus (FTC). Unfortunately, there aren't enough basic rules of integration to integrate anything you can write down like there are for differentiation. So, besides these basic rules of integration, various techniques of integration are needed to integrate more functions. We'll look at these techniques later. For now, let's just look the basic rules.

Properties that directly follow from the definition of integration. These properties are the most elementary ones in the sense that they don't even depend on the FTC, but just on the definition of integrals.

First, there's the trivial observation that the integral of 0 is 0. Slightly less trivial is that fact that the integral of a constant is just that constant times the length of the interval. After all, the area of a rectangle is just the height times the width. As a definite integral this says

$$\int_a^b c\,dx = c(b-a).$$

Next, it's obvious that

$$\int_a^a f(x)\,dx = 0.$$

Now, if an interval $[a,b]$ is broken in the middle at a point c, then the integral over $[a,b]$ is the sum of the integrals over $[a,c]$ and $[b,c]$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In fact, it's not necessary that $a < c < b$; any order will do once we make the definition for "backwards" integration

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Also, linearity of integration directly follows from the definition of integration. An operation is said to be *linear* if it preserves addition, subtraction, and multiplication by constants, and so it preserves any linear combination. For integration, we have the identities

$$\begin{aligned}
\int_a^b (f+g)(x)\,dx &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\
\int_a^b (f-g)(x)\,dx &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\
\int_a^b c\,f(x)\,dx &= c\int_a^b f(x)
\end{aligned}$$

The fundamental theorem of calculus. Although we could go on using the definition of integration to find integrals, as Fermat did, it's much easier to use the FTC. Oresme's FTC, in modern notation, says

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a).$$

It's the primary tool used to integrate functions. Suppose you have an integral $\int_a^b g(x)\,dx$ to evaluate. All you have to do is recognize the integrand g as the derivative of some function f, then evaluate $f(b) - f(a)$, that is, evaluate f between the limits of integration. Any function whose derivative is g is called an *antiderivative* of g, or an *indefinite integral* of g. The notation

$$\int g(x)\,dx = f(x) + C$$

without limits of integration indicates that f is an antiderivative of g, that is $f' = g$, and also that any constant C can be added to f to get another antiderivative.

Integrals of powers x^n, polynomials, and roots. We can use the FTC to integrate a power function $f(x) = x^n$ because we can easily find an antiderivative of x^n, except in the one case when $n = -1$. Since the derivative of $\dfrac{x^{n+1}}{n+1}$ is x^n (except when $n = -1$), therefore,

$$\int_a^b x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_a^b$$

or, as an indefinite integral

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Along with the linearity proprieties mentioned above, that allows us to integrate any polynomial. For example,

$$\int (5x^8 - 4x^7 + 2x^2 - x + 14)\,dx = \frac{5x^9}{9} - \frac{4x^8}{8} + \frac{2x^3}{3} - \frac{x^2}{2} + 14x + C.$$

The power rule for derivatives works for negative and fractional powers as well, and that will allow us to integrate negative and fractional powers. For example,

$$\int (x^{-3} + x^{5/3})\,dx = -\tfrac{1}{2}x^{-2} + \tfrac{3}{8}x^{8/3} + C.$$

The one exception is that we don't yet know the integral of $1/x$. Roots are actually fractional powers, so we can integrate them, too. For example,

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \tfrac{2}{3}x^{3/2} + C.$$

We can now integrate any linear combination of powers and roots such as

$$\int (5x^8 - 7x^{5/3} + 5\sqrt[3]{x} - \frac{3}{x^5} + 7)\,dx = \frac{5x^9}{9} - \frac{21x^{8/3}}{8} + \frac{15x^{4/3}}{4} + \frac{3}{4x^4} + 7x + C.$$

Trig functions. Since we know the derivatives $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$, it follows from the FTC that

$$\begin{aligned}\int \sin x\,dx &= -\cos x + C\\ \int \cos x\,dx &= \sin x + C\end{aligned}$$

We know a couple of other differentiation rules for trig functions, namely, $\frac{d}{dx}\tan x = \sec^2 x$ and $\frac{d}{dx}\sec x = \sec x \tan x$, so we get a couple of other integration formulas $\int \sec^2 x\,dx = \tan x + C$ and $\int \sec x \tan x\,dx = \sec x + C$, but they're not nearly as useful.

The difficulty with integration. Unlike differentiation, there are no product, quotient, and chain rules for integration. That means we can't integrate everything we can write down. The product rule and chain rule for differentiation do give us the two most important techniques of integration, namely the techniques called *integration by parts* and *substitution*, and we'll look at substitution next. As the quotient rule for differentiation follows from the product and chain rules; it won't give us anything new for integration.