# First test answers

Math 122 Calculus III
D Joyce, Clark University

Oct 2011

**Scale.** 85-100 A. 70-84 B. 49-69 C. Median 72.

**Problem 1.** On improper integrals. [30; 15 points each part]

For each improper integral, determine whether or not it converges. If it does converge, give its value.

**a.** $\displaystyle\int_2^\infty \frac{2}{x^2-1}\,dx$

This is an improper integral because the interval of integration, $[2,\infty)$, is infinite.

$$\begin{aligned}\int_2^\infty \frac{2}{x^2-1}\,dx &= \int_2^\infty \left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx \\ &= \ln(x-1)-\ln(x+1)\Big|_2^\infty \\ &= \ln\left(\frac{x-1}{x+1}\right)\Big|_2^\infty\end{aligned}$$

Now, since as $x\to\infty$, $\dfrac{x-1}{x+1}\to 1$, and so $\ln\left(\dfrac{x-1}{x+1}\right)\to 0$. Therefore, the integral evaluates as $0-\ln(1/3)=\ln 3$. Thus, the integral converges to $\ln 3$.

**b.** $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$. (Suggestion: substitution.)

Let $u=\ln$ so that $du=\dfrac{1}{x}\,dx$. Note that as $x\to\infty$, so does $u\to\infty$. Then the integral

$$=\int_{\ln 2}^\infty \frac{1}{u}\,du = \ln u\Big|_{\ln 2}^\infty = \ln\infty - \ln 2.$$

Thus, the integral diverges.

**Problem 2.** On limits. [32; 8 points each] Determine the following limits. If you use l'Hôpital's rule, point out where you use it.

**a.** $\displaystyle\lim_{x\to 1}\frac{\ln x}{1-x}$

This is of the form $\frac{0}{0}$ so we can apply l'Hôpital's rule to get the limit

$$\overset{\text{L'H}}{=}\lim_{x\to 1}\frac{1/x}{-1}=-1$$

**b.** $\displaystyle\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)$

This is of the form $\infty-\infty$. It

$$\begin{aligned}&= \lim_{x\to 1}\frac{x-\ln(1+x)}{x\ln(1+x)} \\ &\overset{\text{L'H}}{=} \lim_{x\to 1}\frac{1-1/(1+x)}{\ln(1+x)+x/(1+x)} \\ &= \lim_{x\to 1}\frac{x}{(1+x)\ln(1+x)+x} \\ &\overset{\text{L'H}}{=} \lim_{x\to 1}\frac{1}{\ln(1+x)+(1+x)/(1+x)+1}=\frac{1}{2}\end{aligned}$$

**c.** $\displaystyle\lim_{x\to 0} x^x$

This is of the indeterminant form $0^0$. Write $x^x$ as $e^{x\ln x}$ and determine the limit of the exponent $x\ln x$. That's of the form $0\cdot\infty$, so rewrite it as a quotient and use l'Hôpital's rule.

$$x\ln x=\frac{\ln x}{1/x}\overset{\text{L'H}}{\cong}\frac{1/x}{-1/x^2}=-x$$

which approaches 0. Since $x\ln x\to 0$, therefore $e^{x\ln x}\to e^0=1$.

**d.** $\displaystyle\lim_{n\to\infty}\frac{2^n+5^n}{3^n+5^n}$

This is of the form $\frac{\infty}{\infty}$. The $5^n$ dominates the numerator and denominator, that is to say, it's much greater than $2^n$ or $3^n$, so divide both the numerator and denominator by it. Then it

$$=\lim_{n\to\infty}\frac{(2/5)^n+1}{(3/5)^n+1}=1$$

**Problem 3.** On proofs of convergence. [20]

Recall that a sequence is said to *converge* to the *limit* $L$, if for each $\epsilon>0$, there is some $N$ such that beyond the $N^{\text{th}}$ term, every term is within $\epsilon$ of $L$. Symbolically, this convergence says

$$\forall\epsilon>0,\exists N,\forall n\geq N,|a_n-L|<\epsilon.$$

Consider the sequence whose $n^{\text{th}}$ term is $a_n=1/2^n$. Its limit $L$ is 0. Let $\epsilon$ be an arbitrary positive number. Find a value of $N$ so that for $n \geq N$ it is the case that $|a_n - L| < \epsilon$. (Note: your answer should express $N$ in terms of $\epsilon$.)

We need to find out how large $n$ has to be so that $1/2^n < \epsilon$. Solve that inequality for $n$. Reciprocating, $2^n > 1/\epsilon$, and taking logs, $n \ln 2 > \ln(1/\epsilon)$, so $n$ needs to be greater than $\dfrac{\ln(1/\epsilon)}{\ln 2}$. Therefore $N = \left\lceil \dfrac{\ln(1/\epsilon)}{\ln 2} \right\rceil$ does it.

**Problem 4.** [20; 4 points each part] True/false. For each sentence write the whole word "true" or the whole word "false". If it's not clear whether it should be considered true or false, you may explain in a sentence if you prefer.

**a.** If the $n^{\text{th}}$ term of a sequence $a_n$ lies between $2 \pm 1/n$, then $\lim\limits_{n\to\infty} a_n = 2$. True by the sandwich theorem.

**b.** If $a_n \to 0$ and $b_n \to 1$, then $a_n^{b_n} \to 0$. True; the determinant form $0^1$ does equal 0.

**c.** If the terms of a sequence are decreasing and bounded above, then the sequence converges. False; $0, -1, -2, -3, \ldots$ diverges to $-\infty$.

**d.** If $a_n \to \pi/2$, then $\tan a_n$ is a divergent sequence. True; $\tan \frac{\pi}{2} = \pm\infty$.

**e.** If $\dfrac{a_n}{b_n} \to 3$ and $b_n \to 0$, then $a_n \to 0$. True; if the quotient converges but the denominator approaches 0, then the numerator also has to approach 0.