Second test answers

Math 122 Calculus III
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Scale. 85-100 A. 68-84 B. 49-67 C. Median 68

Problem 1. On curves in polar coordinates. [10]

Convert the equation $r = \sin\theta + \sin 2\theta$ of a curve given in polar coordinates to one in rectangular coordinates, x and y.

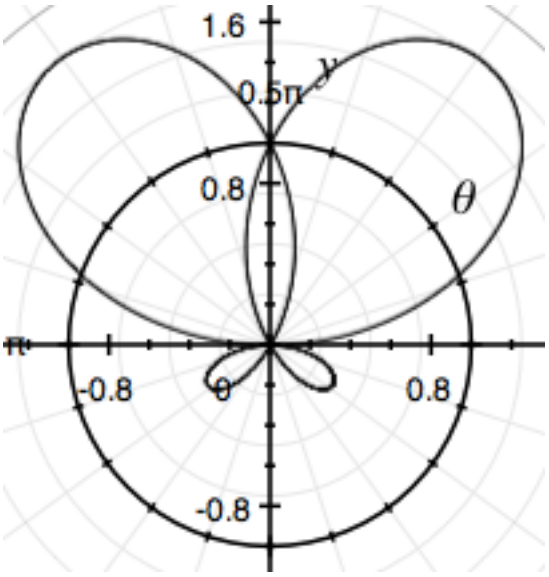
The relevant equations to do the conversion are $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, and $\sin 2\theta = 2\sin\theta\cos\theta$. Using the last, we have

$$r = \sin\theta + 2\sin\theta\cos\theta$$

so $r^3 = r^2\sin\theta + 2r^2\sin\theta\cos\theta$. That gives $r^2 = ry + 2xy$, so we have

$$x^2 + y^2 = y\sqrt{x^2 + y^2} + 2xy.$$

With a little algebra, you can convert this to the equation $(x-y)^4 = y^2(x^2+y^2)$. In case you're wondering, the graph for this equation looks like



Problem 2. On areas in polar coordinates. [10]

Write down an integral which gives the area inside the curve $r = \sin\theta + \sin 2\theta$ between the angles $\theta = 0$ and $\theta = \pi/4$. Do not evaluate the integral.

$$\int_\alpha^\beta \frac{1}{2}r^2\,d\theta = \int_0^{\pi/4} \frac{1}{2}(\sin\theta + \sin 2\theta)^2\,d\theta$$

(The area in question is half of the first large leaf of the graph.)

Problem 3. On the concept of convergence of a series. [21; 7 points each part]

a. Complete the definition of convergence. The series $a_1 + a_2 + \cdots + a_n + \cdots$ converges to the sum S if

the limit of the partial sums S_n is S, $\lim_{n\to\infty} S_n = S$, where the n^{th} partial sum S_n is $a_1 + a_2 + \cdots + a_n$.

b. Consider the geometric series $\frac{4}{5} + \left(\frac{4}{5}\right)^2 + \cdots + \left(\frac{4}{5}\right)^n + \cdots$. Evaluate the n partial sum $S_n = \frac{4}{5} + \left(\frac{4}{5}\right)^2 + \cdots + \left(\frac{4}{5}\right)^n$

$$S_n = a\frac{1-r^{n+1}}{1-r} = \frac{4}{5}\frac{1-(4/5)^n}{1-4/5} = 4(1-(4/5)^n)$$

c. Use what you found in part **b** and the definition of convergence to show that the series in part **b** converges to the sum $S = 4$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 4(1-(4/5)^n) = 4$$

Problem 4. On convergence of series. [40; 10 points each] Determine whether the following series converge or diverge. Name any convergence tests you use and show any relevant computations.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{3/2}}$

This is 3 times a p-series with $p = 3/2$, so it converges. There are several convergence tests which will also show it converges.

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n^2-1}{4n^2+5n+2}$

Since $\dfrac{3n^2-1}{4n^2+5n+2} \to \dfrac{3}{4}$, the terms don't approach 0. Therefore, the series diverges.

c. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2-1}{4n^4+5n+2}$

Either a limit comparison test or a regular comparison test of this series to the series $\sum 1/n^2$ will show that it converges.

d. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{20}}{10^n}$

The ratio test works well here.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{20}}{10^{n+1}}\,\frac{10^n}{n^{20}} = \left(\frac{n+1}{n}\right)^{20}\frac{1}{10} \to \frac{1}{10}$$

Since this ratio approaches a number less than 1, it converges.

Problem 5. On power series. [20; 10 points each part]

a. Determine the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} nx^n$.

The limit of the ratio a_{n+1}/a_n of the coefficients approaches the reciprocal of the radius of convergence.

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \to 1$$

so the radius of convergence is 1.

The endpoints of the interval are $x = \pm 1$. Both $\sum n$ and $\sum(-1)^n n$ diverge. Therefore, the interval of convergence is $(-1, 1)$.

b. Suppose a function f has the following derivatives when evaluated at 0. $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 3$, and in general $f^{(n)}(0) = n$. Write down the only power series that represents f.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^n \\ &= x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots \end{aligned}$$

since $a_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{n}{n!} = \dfrac{1}{(n-1)!}$.

By the way, the function here is $f(x) = xe^x$.