Math 122 Calculus III

Second Test Answers, November 2012

Scale. 88–100 A, 75–87 B, 55–74 C. Median 90.

1. [12] Determine the sum of the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$

This is a geometric series. The first term in $a = \frac{2}{3}$ and the ratio is $r = \frac{2}{3}$. Therefore, the sum is $S = \frac{a}{1-r} = \frac{2/3}{1/3} = 2$.

2. [40; 10 points each part] On convergence of series with positive terms. For each series, apply one or more convergence tests to determine whether the series converges. Be sure to mention which test you use.

a. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

A comparison test or a limit comparison test to the divergent series $\sum \frac{1}{n}$ shows this diverges. Since $\frac{1}{\ln n} > \frac{1}{n}$, and the series $\sum \frac{1}{n}$ diverges, so does the series $\frac{1}{\ln n}$ diverge.

b. $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

The ratio test works well here

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!} \frac{n!}{10^n} = \frac{10}{(n+1)} \to 0$$

Since this limit is less than 1, therefore the series converges.

c. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. Suggestion: integral test.

The integral test is the only test that will work here. You can evaluate the integral

$$\int^{\infty} \frac{dx}{x(\ln x)^2}$$

with the substitution $u = \ln x$, $du = \frac{dx}{x}$. Note that as $x \to \infty$, so $u \to \infty$ also, so the integral becomes

$$\int^{\infty} \frac{du}{u^2} = \left. \frac{-1}{u} \right|^{\infty} = \frac{1}{\infty} = 0$$

so the integral converges. Therefore, the series also converges.

d. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 2n + 5}$

You could use the comparison test or the limit comparison test to compare this series to the convergent p-series $\sum b_n = \sum \frac{1}{n^3}$. Here's the limit comparison test

$$\frac{a_n}{b_n} = \frac{1}{n^3 + 2n + 5} \frac{n^3}{1} = \frac{n^3}{n^3 + 2n + 5} \to 1$$

Since this limit is a finite number, and the series $\sum b_n$ converges, so does the series $\sum a_n = \sum \frac{1}{n^3 + 2n + 5}$.

Problem 3. [16; 4 points each part] True/false.

a. If the terms of a series approach 0, then the series converges. *False.* The harmonic series $\sum \frac{1}{n}$ gives a counterexample.

b. If a power series represents a function, then you can find a power series to represent the derivative of the function by differentiating each term, that is,

$$\text{if } f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ then } f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

True. You can differentiate a series term by term.

c. For an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, if $\lim_{n\to\infty} a_n = 0$, then the series converges. *False.* The terms need to decrease to 0 to apply the alternating series test. A counterexample is

$$\frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} + \cdots$$

whose terms approach 0 but do not decrease to 0. It diverges.

d. For an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, if $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. *True.* If the terms don't approach 0, the series diverges whether it's alternating or not.

Problem 4. [16] On power series. Consider the power series

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{1}{n2^n} x^n = \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{24}x^3 + \cdots + \frac{1}{n2^n}x^n + \cdots$$

Determine the radius of convergence, r, for this series.

Consider the limit of the ratio of the coefficients.

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)2^{n+1}} \frac{n2^n}{1} = \frac{n2^n}{(n+1)2^{n+1}} \to \frac{1}{2}.$$

The reciprocal of this limit is the radius of convergence $r = 2$.

Problem 5. [16] For the function $f(x) = \arcsin x$, determine the terms of the power series up to x^3, that is, find a_0 through a_3 in the power series $a_0+a_1x+a_2x^2+a_3x^3+\cdots$. To save you time, here are the first few derivatives of $\arcsin x$.

$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}$
$\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{x}{(1-x^2)^{3/2}}$	$\dfrac{2x^2+1}{(1-x^2)^{5/2}}$	$\dfrac{6x^3+9x}{(1-x^2)^{7/2}}$

Since $a_n = \dfrac{f^{(n)}(0)}{n!}$, therefore

$$\begin{aligned} a_0 &= \frac{1}{0!} \arcsin 0 = 0 \\ a_1 &= \frac{1}{1!} \frac{1}{\sqrt{1-0^2}} = 1 \\ a_2 &= \frac{1}{2!} \frac{0}{(1-0^2)^{3/2}} = 0 \\ a_3 &= \frac{1}{3!} \frac{2(0^2)+1}{(1-0^2)^{5/2}} = \frac{1}{6} \end{aligned}$$

Therefore the power series for $\arcsin x$ begins

$$\arcsin x = x + \frac{1}{6}x^3 + \cdots$$

(Note that since $\arcsin x$ is an odd function, the coefficients of all the even powers of x will be 0.)