# Cauchy's Inequality
Math 122 Calculus III

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**Dot products in $n$-space.** The dot product is an operation $\cdot : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ that takes two vectors $\mathbf{v}$ and $\mathbf{w}$ and gives a scalar $\mathbf{v}\cdot\mathbf{w}$ by adding the products of corresponding elements, that is,

$$\mathbf{v} \cdot \mathbf{w} = (v_1, v_2, \ldots, v_n) \cdot (w_1, w_2, \ldots, w_n) = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

The square of the length of the vector is a dot product:

$$\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}.$$

The dot product in $n$-space has all the usual properties that the dot product for 2-space has. It's commutative, and it's linear in both coordinates.

**Dot product and angles in $n$-space.** Just as in the $n = 2$ dimensional case, the law of cosines still applies (but it has to be used in the plane formed by the two vectors) to show

$$\mathbf{w} \cdot \mathbf{v} = \|\mathbf{v}\|\,\|\mathbf{w}\| \cos\theta,$$

equivalently,

$$\cos\theta = \frac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{v}\|\,\|\mathbf{w}\|},$$

where $\theta$ is the angle between the two vectors.

Note that since $\cos\theta$ is between $\pm 1$, therefore the absolute value of

$$\frac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

is less than or equal to 1. Hence,

$$|\mathbf{w} \cdot \mathbf{v}| \leq \|\mathbf{w}\|\,\|\mathbf{v}\|.$$

This last inequality is called the *Cauchy inequality*. More on it below.

As in the 2-dimensional case, two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal (also called perpendicular), written $\mathbf{v} \perp \mathbf{w}$, if and only if $\mathbf{w} \cdot \mathbf{v} = 0$.

**The triangle inequality and the Cauchy inequality in $n$-space.** The triangle inequality says

$$\|\mathbf{w} - \mathbf{v}\| \leq \|\mathbf{w}\| + \|\mathbf{v}\|,$$

or, replacing $-\mathbf{v}$ by $+\mathbf{v}$, it says

$$\|\mathbf{w} + \mathbf{v}\| \leq \|\mathbf{w}\| + \|\mathbf{v}\|.$$

Let's see if we can prove it algebraically.

Since all the quantities are nonnegative, the last inequality is logically equivalent to

$$\|\mathbf{w} + \mathbf{v}\|^2 \leq (\|\mathbf{w}\| + \|\mathbf{v}\|)^2.$$

We can rewrite the left hand side as

$$\begin{aligned} \|\mathbf{w} + \mathbf{v}\|^2 &= (\mathbf{w} + \mathbf{v}) \cdot (\mathbf{w} + \mathbf{v}) \\ &= \mathbf{w} \cdot \mathbf{w} + 2\mathbf{w} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{w}\|^2 + 2\mathbf{w} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \end{aligned}$$

and we can rewrite the right hand side as

$$(\|\mathbf{w}\| + \|\mathbf{v}\|)^2 = \|\mathbf{w}\|^2 + 2\|\mathbf{w}\| \, \|\mathbf{v}\| + \|\mathbf{v}\|^2.$$

Thus, the triangle inequality with a sum,

$$\|\mathbf{w} + \mathbf{v}\| \leq \|\mathbf{w}\| + \|\mathbf{v}\|,$$

is logically equivalent to

$$\mathbf{w} \cdot \mathbf{v} \leq \|\mathbf{w}\| \, \|\mathbf{v}\|.$$

Likewise, the triangle inequality with a difference,

$$\|\mathbf{w} - \mathbf{v}\| \leq \|\mathbf{w}\| + \|\mathbf{v}\|,$$

is logically equivalent to

$$-\mathbf{w} \cdot \mathbf{v} \leq \|\mathbf{w}\| \, \|\mathbf{v}\|.$$

We conclude that together they are logically equivalent to the Cauchy inequality

$$|\mathbf{w} \cdot \mathbf{v}| \leq \|\mathbf{w}\| \, \|\mathbf{v}\|,$$

which is also known as the Cauchy-Schwarz inequality or as Bunyakovsky's inequality. Cauchy mentioned this inequality in 1821. The others generalized it.

If you accept the statement above that the law of cosines applies in dimension $n$, then we've already proved the Cauchy inequality, but we can prove it directly without resort to $n$-dimensional geometry.

Here's a clear, direct proof for the case $n = 3$. The argument works for general $n$.

In order to prove the Cauchy inequality, we'll prove its square instead

$$(\mathbf{w} \cdot \mathbf{v})^2 \leq \|\mathbf{w}\|^2 \, \|\mathbf{v}\|^2.$$

That says

$$(v_1w_1 + v_2w_2 + v_3w_3)^2 \leq (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2),$$

which, when expanded, says

$$\left.\begin{array}{clclcl} & v_1w_1v_1w_1 & + & v_1w_1v_2w_2 & + & v_1w_1v_3w_3 \\ + & v_2w_2v_1w_1 & + & v_2w_2v_2w_2 & + & v_2w_2v_3w_3 \\ + & v_3w_3v_1w_1 & + & v_3w_3v_2w_2 & + & v_3w_3v_3w_3 \end{array}\right\} \leq \left\{\begin{array}{clclcl} & v_1^2w_1^2 & + & v_1^2w_2^2 & + & v_1^2w_3^2 \\ + & v_2^2w_1^2 & + & v_2^2w_2^2 & + & v_2^2w_3^2 \\ + & v_3^2w_1^2 & + & v_3^2w_2^2 & + & v_3^2w_3^2 \end{array}\right.$$

Now, some of the terms are the same, so they can be eliminated to get

$$\left.\begin{array}{clclcl} & & & v_1w_1v_2w_2 & + & v_1w_1v_3w_3 \\ + & v_2w_2v_1w_1 & & & + & v_2w_2v_3w_3 \\ + & v_3w_3v_1w_1 & + & v_3w_3v_2w_2 & & \end{array}\right\} \leq \left\{\begin{array}{clclcl} & & & v_1^2w_2^2 & + & v_1^2w_3^2 \\ + & v_2^2w_1^2 & & & + & v_2^2w_3^2 \\ + & v_3^2w_1^2 & + & v_3^2w_2^2 & & v_3^2w_3^2 \end{array}\right.$$

Note that each remaining term on the left appears twice. We can move all the terms to the right hand side to get the equivalent inequality

$$0 \leq \left\{\begin{array}{clclcl} & v_1^2w_2^2 & - & 2v_1w_2v_2w_1 & + & v_2^2w_1^2 \\ + & v_1^2w_3^2 & - & 2v_1w_3v_3w_1 & + & v_3^2w_1^2 \\ + & v_2^2w_3^2 & - & 2v_2w_3v_3w_2 & + & v_3^2w_2^2 \end{array}\right.$$

which we can rewrite as

$$0 \leq \left\{\begin{array}{r} (v_1w_2 - v_2w_1)^2 \\ +(v_1w_3 - v_3w_1)^2 \\ +(v_2w_3 - v_3w_2)^2 \end{array}\right.$$

which is obviously true, since a sum of squares is always greater than or equal to 0. Since all these inequalities are logically equivalent, and the last is true, therefore they are all true. Thus, we have proved the Cauchy inequality.

Math 122 Home Page at `http://math.clarku.edu/~djoyce/ma122/`