Completeness of **R**
Math 122 Calculus III
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This is a summary of the main points we've discussed in class about the completeness of the real numbers. This concept is needed for the mathematical foundations for sequences and series.

**Definition of LUBs, etc.** Our discussion of the formal mathematics began with the definitions of least upper bounds and the least upper bound axiom.

**Definition 1.** An *upper bound* of a set $S$ of real numbers is any real number which is greater or equal to all numbers in $S$. A *lower bound* is any which is less than or equal to all numbers in $S$. A *least upper bound* is an upper bound which is less than or equal to all upper bounds. A *greatest lower bound* is a lower bound which is greater than or equal to all lower bounds.

Note that this definition does not say that any of these things exist. Sometimes they do and sometimes they don't. It depends on what set $S$ is.

We'll use the notation $\operatorname{lub} S$ for the least upper bound of $S$, should it exist, and $\operatorname{glb} S$ for the greatest lower bound, should it exist. The least upper bound is also called the *supremum* of the set, and the greatest lower bound is also called the *infemum* of the set.

Here are some examples of sets and their bounds.

**Example 2** (a finite set)**.** $S = \{2, 8, 12\}$. Any number $\geq 12$ is an upper bound. The least upper bound is 12. Any number $\leq 2$ is a lower bound. The greatest lower bound is 2. For a finite set $S$, $\operatorname{lub} S$ is always the largest number in $S$, and $\operatorname{glb} S$ is always the smallest number in $S$.

**Example 3** (a closed interval)**.** $S = [4, 9]$. $\operatorname{lub} S = 9$, $\operatorname{glb} S = 4$. Like in the previous example, the lub and the glb are the largest and smallest numbers in the set. Any time $S$ contains a largest number, that number is its lub. Likewise, any time $S$ contains a smallest number, that's its glb.

**Example 4** (a bounded open interval)**.** $S = (4, 9)$. Same lub and glb as in the last example, but this time they aren't numbers in $S$.

**Example 5.** $S = (-\infty, 9)$. Same lub as the previous example. This $S$ has no lower bounds at all, so it has no greatest lower bound.

**Example 6.** $S = \{0.9, 0.99, 0.999, \ldots\}$. This is an infinite set. The smallest number in $S$ is 0.9, so that's its glb. This $S$ contains no largest number, but 1 is the least upper bound of $S$.

Although some sets don't have lub's and glb's, a set can have at most one of each. You can't have two different numbers being lub's since each would have to be greater than or equal to the other, hence equal.

**The completeness axiom.** There are various different logically equivalent statements that can be used as an axiom of the completeness of the real numbers. We'll use one called the least upper bound axiom.

**Axiom 7** (Least upper bound axiom)**.** Each nonempty set of real numbers that has an upper bound has a least upper bound.

**Theorem 8.** Each nonempty set of real numbers that has a lower bound has a greatest lower bound.

*Proof.* Let $S$ bet a set of real numbers that has a lower bound $L$. Let $T$ be the set $T = \{x \mid -x \in S\}$. Then $-L$ is an upper bound of $T$. (For if $x \in T$, then $-x \in S$, so $L \leq -x$, so $s \leq -L$.) Therefore, by the lub axiom, $T$ has a least upper bound $M$. Then $-M$ is a greatest lower bound of $S$. (Details omitted.) Q.E.D.

**Theorem 9.** If $M = \operatorname{lub} S$ and $\epsilon > 0$, then there exists at least one $x \in S$ such that $M - \epsilon < x \leq M$.

*Proof.* If there weren't such an $x$, then $M - \epsilon$ would be a smaller upper bound for $S$ than $M$. Q.E.D.

**The IVT and the EVT.** The Intermediate Value Theorem (IVT) and the Extremal Value Theorem (EVT) for continuous functions follow from the least upper bound axiom. These are things you probably studied in the first semester of calculus. We won't prove them here, but it's worthwhile reviewing them.

**Theorem 10** (IVT)**.** If $f$ is a continuous function on $[a, b]$ then it takes on all values between $f(a)$ and $f(b)$, that is, if $K$ lies between $f(a)$ and $f(b)$, then there is some $c$ between $a$ and $b$ such that $f(c) = K$.

**Theorem 11** (EVT)**.** If $f$ is a continuous function on $[a, b]$ then it takes on a maximum value and a minimum value.

The EVT is used to prove the Mean Value Theorem (MVT) which says that if a function $f$ is differentiable on and interval $[a, b]$ average rate of change $\dfrac{f(b) - f(a)}{b - a}$ over that interval is the value of the derivative somewhere on that interval, that is, there is some $c$ in that interval so that $f'(c)$ is equal to that average rate of change.

The MVT is used in turn to prove that such statements as functions with 0 derivatives are constant, functions with positive derivatives are increasing, etc.

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