Determinants
Math 122 Calculus III
D Joyce, Fall 2012

What they are. A determinant is a value associated to a square array of numbers, that square array being called a square matrix. For example, here are determinants of a general 2×2 matrix and a general 3×3 matrix.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi.$$

The determinant of a matrix A is usually denoted $|A|$ or det (A).

You can think of the rows of the determinant as being vectors. For the 3×3 matrix above, the vectors are $\mathbf{u} = (a, b, c)$, $\mathbf{v} = (d, e, f)$, and $\mathbf{w} = (g, h, i)$. Then the determinant is a value associated to n vectors in $\mathbf{R}^n$.

There's a general definition for $n \times n$ determinants. It's a particular signed sum of products of n entries in the matrix where each product is of one entry in each row and column. The two ways you can choose one entry in each row and column of the 2×2 matrix give you the two products ad and bc. There are six ways of chosing one entry in each row and column in a 3×3 matrix, and generally, there are $n!$ ways in an $n \times n$ matrix. Thus, the determinant of a 4×4 matrix is the signed sum of 24, which is 4!, terms. In this general definition, half the terms are taken positively and half negatively. In class, we briefly saw how the signs are determined by permutations.

What they're used for. You've probably already seen how determinants can by used to solve a system of n linear equations in n unknowns. Cramer's rule can do that.

More generally, determinants can be used any time there are linear equations and in many ways. They're central tools in the whole subject of linear algebra.

As linear algebra is used throughout mathematics and science, determinants get a lot of use.

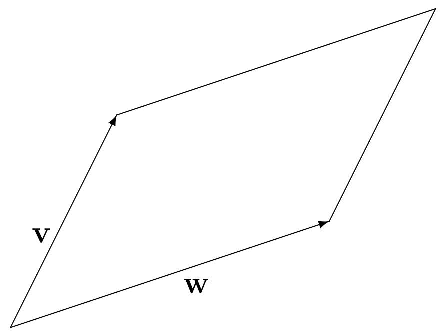
For instance, in the subject of differential equations, determinants appear in the solution of systems of linear differential equations. An example of such is

$$\begin{aligned} x' &= 3x - 4y + z \\ y' &= x - 2y + 3z \\ z' &= x - 3y + 4z \end{aligned}$$

Another is the one whose solutions include sines and cosines, $y'' = -y$. The determinant for a system of linear differential equations is called the Wronskian.

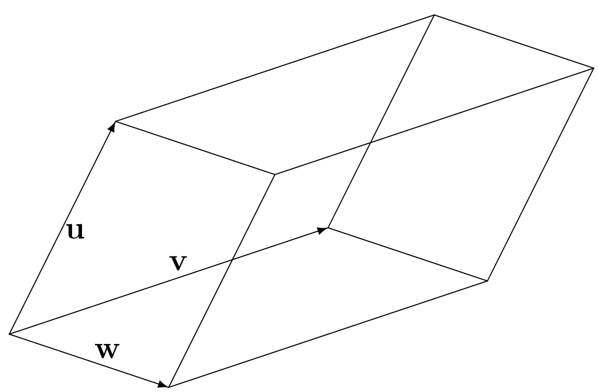
Differential equations like this appear in a lot of models in physics, chemistry, economics, and other physical and social sciences. Linear dynamical systems have a lot of applications.

Determinants also have a geometrical interpretation. In two dimensions, the determinant gives the signed area of a parallelogram. If **v** and **w** are two vectors with their tails at the same point, then they form two sides of a parallelogram.



The signed area of the parallelogram is the value of the 2×2 matrix whose rows are **v** and **w**. The sign includes the orientation of the parallelogram. If **v** and **w** are exchanged, the signed area is negated.

In three dimensions, the determinant gives the signed volume of a parallelepiped whose edges are **u**, **v**, and **w**, the three rows of the matrix.



Likewise, in higher dimensions, the determinant gives the signed n-dimensional volume of an n-dimensional parallelepiped.

This gives determinants and important role to play anytime area or volume is under consideration, and that means things involving integrals. In particular, the substitution rule in multivariate uses a determinant called the Jacobian.

Cramer's rule. We're not particularly interested in Cramer's rule, but it's worthwhile discussing it for the historical reasons if nothing else. It was one of the first uses of determinants. It is a method to find the solution to a system of n equations in n unknowns when there is exactly one solution. The solution is has the determinant in the denominator, and the only time the determinant is not zero is when there's a unique solution.

Here's an example to show how to apply Cramer's rule. Let's suppose we have the following system of three equations in three unknowns.

$$\begin{array}{rcrcrcr} x & + & y & + & 3z & = & 6 \\ 2x & + & 3y & - & 4z & = & -2 \\ 3x & - & 2y & + & 5z & = & 7 \end{array}$$

First, compute the determinant Δ of the 3×3 coefficient matrix.

$$\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & 5 \end{vmatrix} = -54$$

Next, replace the first column by the constant vector, and compute that determinant.

$$\Delta_x = \begin{vmatrix} 6 & 1 & 3 \\ -2 & 3 & -4 \\ 7 & -2 & 5 \end{vmatrix} = -27$$

Then in the unique solution, $x = \Delta_x/\Delta = \frac{1}{2}$. Next, replace the second column by the constant vector, and compute that determinant.

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & -2 & -4 \\ 3 & 7 & 5 \end{vmatrix} = -54$$

So $y = \Delta_y/\Delta = 1$. Likewise, replace the third column by the constant vector.

$$\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 3 & -2 \\ 3 & -2 & 7 \end{vmatrix} = -81$$

which gives $z = \frac{3}{2}$. Thus, the unique solution is $(x, y, z) = (\frac{1}{2}, 1, \frac{3}{2})$.

The determinant is characterized by it's properties. If you think of an $n \times n$ determinant as being a function f of n n-dimensional vectors, it has certain properties that characterize it. It's the unique multilinear, alternating function whose value is 1 when applied to the standard basis vectors.

Multilinear means it's linear in each coordinate. Here's what that says for the first coordinate. If $\mathbf{v}_1 = \mathbf{u} + \mathbf{w}$, then

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = f(\mathbf{u}, \mathbf{v}_2, \ldots, \mathbf{v}_n) + f(\mathbf{w}, \mathbf{v}_2, \ldots, \mathbf{v}_n),$$

and if $\mathbf{v}_1 = k\mathbf{u}$ where k is a scalar, then

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = kf(\mathbf{u}, \mathbf{v}_2, \ldots, \mathbf{v}_n).$$

Determinants have that property.

Alternating means that if you exchange any two arguments, then the value of the function is negated. For instance, if the first two arguments are exchanged, then

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = -f(\mathbf{v}_2, \mathbf{v}_1, \ldots, \mathbf{v}_n).$$

Finally, $f(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = 1$, where $\mathbf{e}_i$ is the i^{th} standard basis vector $(0, \ldots, 1, \ldots, 0)$ with a 0 in every coordinate except the i^{th} coordinate which is a 1. The matrix here has 1's down the main diagonal and 0's elsewhere. It's called the *unit matrix* and denoted $\mathbf{I}$.

Each of these three properties can be interpreted geometrically. The last one says the n-dimensional unit n-cube has 1 as its n-dimensional volume. (So when $n = 2$ it says the unit square has area 1.)

The second property says that the volume of a parallelepiped is multiplied by a factor of k when you stretch the parallelepiped in one direction by a factor of k.

The first is a little more complicated to visualize, but says something like if you stack two parallelepipids with the same base, one on the other, then parallelepiped between the opposite bases has an n-dimensional volume which is the sum of the other two.

The theorem here is that for each dimension n, there is one and only one function with these three properties, and it's the determinant.

More properties of determinants. There are some other properties that follow from the characterizing properties.

Any matrix with two identical rows has 0 determinant.

Proof: Interchange those two rows to negate its determinant, but since you get the same determinant, that determinant has to be its own negation. The only number which is its own negation is 0. Q.E.D.

If one row is a multiple of another, then the determinant is 0.

Proof: By linearity, you can factor out that multiple to get a matrix with two equal rows, and that has determinant 0, so the determinant of the original matrix is a constant times 0, that is, 0. Q.E.D.

If you change a matrix by adding a multiple of one row to another row, then the determinant doesn't change.

Proof: This follows from the multilinearity property described above. The new matrix is formed from the original matrix and another matrix where one row is a multiple of another row, hence has 0 determinant. For instance, if you add twice the first row to the second, you can interpret the resulting matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + 2a_{11} & a_{22} + 2a_{12} & a_{23} + 2a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

as being created from these two matrices with the all same rows but the middle one

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{11} & 2a_{12} & 2a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and the second matrix has a 0 determinant because one row is a multiple of another. Therefore, the resulting matrix has the same determinant as the first matrix. Q.E.D.

There are some other useful properties, most of them easy to show. The one exchanging rows and columns is more difficult.

If a matrix has a row of zeros, then its determinant is 0.

If you exchange all the rows and columns, that is, flip it across the major diagonal, then the determinant is unchanged.

A diagonal matrix, that is, one whose entries off the major diagonal are all 0, has a determinant equal to the product of its diagonal entries.

An upper-triangular matrix, that is, one whose entries below the major diagonal are all 0, also has a determinant equal to the product of its diagonal entries.. (Likewise for lower-triangular matrices.)

Calculating determinants. For 2×2 determinants and 3×3 determinants, the definition given above works fine. For larger determinant and even for 3×3 determinants, it's better to use the properties.

You use the properties to clear out enough entries of the matrix to make it an upper-triangular matrix, then the determinant is just the product of the diagonal entries.

We went through a couple of examples in class. What you do is you choose one non-zero entry, called the *pivot* in each column and work to clear out other entries in that column by subtracting the appropriate multiple of the row containing that pivot from the other row.

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