Series Convergence Tests
Math 122 Calculus III

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Some series converge, some diverge.

Geometric series. We've already looked at these. We know when a geometric series converges and what it converges to. A geometric series $\sum_{n=0}^{\infty} ar^n$ converges when its ratio r lies in the interval $(-1,1)$, and, when it does, it converges to the sum $\dfrac{a}{1-r}$.

The harmonic series. The standard harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges to ∞. Even though its terms $1, \frac{1}{2}, \frac{1}{3}, \ldots$ approach 0, the partial sums S_n approach infinity, so the series diverges.

The main questions for a series.

Question 1: given a series does it converge or diverge?

Question 2: if it converges, what does it converge to?

There are several tests that help with the first question and we'll look at those now.

The term test. The only series that can converge are those whose terms approach 0. That is, if $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$.

Here's why. If the series converges, then the limit of the sequence of its partial sums approaches the sum S, that is, $S_n \to S$ where S_n is the n^{th} partial sum $S_n = \sum_{k=1}^{n} a_k$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

The contrapositive of that statement gives a test which can tell us that some series diverge.

Theorem 1 (The term test)**.** If the terms of the series don't converge to 0, then the series diverges.

Note, however, the terms converging to 0 doesn't imply the series converges, as the harmonic series gives a counterexample to that.

The term test can be used to show that the following series don't converge

$$\sum \frac{n}{n+1} \qquad \sum \frac{n!}{n^2} \qquad \sum (-1)^n \frac{n}{2n+1}$$

because their terms do not approach 0.

The rest of the tests in this note deal with positive series, that is, a series none of whose terms are negative. Note that the only way a positive series can diverge is if it diverges to infinity, that is, its partial sums approach infinity.

The comparison test. Essentially, a positive series with smaller terms sums to a smaller number than a series with larger terms.

Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are two positive series and every term of the first is less than or equal to the corresponding term of the second, that is, $a_n \leq b_n$ for all n. If that's the case, we'll say the second series *dominates* the first series. Then each partial sum $S_n = \sum_{k=1}^{n} a_k$ of the first series is less than or equal to the corresponding partial sum $T_n = \sum_{k=1}^{n} b_k$ of the second series. Furthermore, since these are both positive series, so both sequences $\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ of partial sums are increasing sequences.

Suppose the second sequence converges to a number T, that is, $T_n \to T$. Then T bounds the sequence $\{T_n\}_{n=1}^{\infty}$, so it also bounds the smaller sequence $\{S_n\}_{n=1}^{\infty}$. Since an increasing bounded sequence has a limit, therefore $\{S_n\}_{n=1}^{\infty}$ has a limit, S, and $S \leq T$.

Thus, we've proven the following theorem. The second statement is the contrapositive of the first, so it's also true.

Theorem 2 (The comparison test)**.** Suppose that one positive series is dominated by another. If the second converges, then so does the first. If the first diverges to infinity, then so does the second.

You can think of this theorem as simply saying that

$$\boxed{\text{If } a_n \leq b_n \text{ for each } n, \text{ then } \sum a_n \leq \sum b_n}$$

Example 3. Any series dominated by a positive convergent geometric series converges. For instance, we'll show $\sum_{n=4}^{\infty} \frac{1}{n!}$ converges since it's dominated by the convergent geometric series $\sum_{n=4}^{\infty} \frac{1}{2^n}$. All we need to do is show that $\frac{1}{n!} \leq \frac{1}{2^n}$ for large n. But for $n \geq 4$, $2^n \leq n!$. Thus $\sum_{n=4}^{\infty} \frac{1}{n!}$ is dominated by a convergent geometric series, and, so, it's also a convergent series. Since the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ only has finitely many more terms, it also converges.

Note that whether a series converges or diverges doesn't at all depend on the first few terms. It only depends the rest of them, the "tail" of the series. For that reason, when we're only interested in convergence, we'll leave usually abbreviate our sigma notation and say $\sum \frac{1}{n!}$ converges.

Example 4. Since $\dfrac{1}{n} < \dfrac{1}{\ln n}$ (for $n \geq 2$), and $\sum \dfrac{1}{n}$ diverges, so does $\sum \dfrac{1}{\ln n}$.

The integral test. For many positive series, the question of convergence for the series can be replaced by a question of convergence for a closely related integral. We'll illustrate this with an example first.

Example 5. Consider the series

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \cdots + \tfrac{1}{n^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We can't use the comparison test on this since we don't yet know any divergent series that it dominates, and we don't know any convergent series that it dominates. But we can extend $1/n^2$ to a nice continuous function $f(x) = 1/x^2$ and look at the integral $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$.

We'll draw a figure in class that shows the following inequality

$$\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The first sum is a rectangular underestimate for the integral, and the second sum is an overestimate. Thus, our series sums to almost the integral, the error being at most the value of the first term. We can easily evaluate that improper integral.

$$\int_1^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^{\infty} = 1$$

Since the integral converges, so does the series. We also get crude bounds on the sum of the series, namely, it sums to a value between 1 and 2.

The exact sum of this series is difficult to find. We won't find it in this course, but it turns out to be $\pi^2/6$.

We can generalize this example to prove the following theorem. The only special property of the function $f(x)$ that we needed was that it was a decreasing function.

Theorem 6 (The integral test)**.** If f is a decreasing positive function defined on $[1, \infty)$, then the series $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges if and only if the integral $\displaystyle\int_1^{\infty} f(x)\,dx$ converges. In that case, difference between the sum of the series and the value of the integral is at most $f(1)$.

It's useful to have a notation to indicate that two series or integrals either both converge or both diverge. We'll use a tilde, $\sim$, for that. Then the integral test can be summarized as

$$\boxed{\text{If } f \text{ is positive and decreasing then } \sum f(n) \sim \int f(x)\,dx}$$

p-series Series of the form $\sum \frac{1}{n^p}$, where p is a constant power, are called p-series. When $p = 1$, the p-series is the harmonic series which we know diverges. When $p = 2$, we have the convergent series mentioned in the example above. By use of the integral test, you can determine which p-series converge.

Theorem 7 (p-series)**.** A p-series $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.

Proof. If $p \leq 1$, the series diverges by comparing it with the harmonic series which we already know diverges. Now suppose that $p > 1$. The function $f(x) = 1/x^p$ is a decreasing function, so to determine the convergence of the series we'll detemine the convergence of the corresponding integral.

$$\int_1^\infty \frac{1}{x^p} = \left.\frac{-1}{(p-1)x^{p-1}}\right|_1^\infty = -0 + \frac{1}{p-1}$$

Since the integral converges, so does the series. Q.E.D.

Some example divergent p-series are $\sum \frac{1}{n}$ and $\sum \frac{1}{\sqrt{n}}$. Some convergent ones are $\sum \frac{1}{n^2}$, $\sum \frac{1}{n\sqrt{n}}$, and $\sum \frac{1}{n^{1.001}}$.

The limit comparison test. This test is an improvement on the comparison test. It incorporates the fact that a series converges if and only if a constant multiple of it converges (provided that constant is not 0, of course). So long as you can compare a multiple of one series to another, that's enough to do a comparison.

Theorem 8 (Limit comparison test)**.** Given two positive series $\sum a_n$ and $\sum b_n$ where the ratio of their terms a_n/b_n approaches a positive number, then they either both converge or diverge, that is, $\sum a_n \sim \sum b_n$.

Proof. Suppose that $a_n/b_n \to L$, a positive number. Let $\epsilon = L/2$. Then for large n, $|a_n/b_n - L| < L/2$. Hence, $\frac{L}{2}b_n \leq a_n \leq \frac{3L}{2}b_n$. Now, if $\sum a_n$ converges, then by the comparison test, so does $\sum \frac{L}{2}b_n$ converge, hence $\sum b_n$ converges. On the other hand, if $\sum b_n$, converges, so does $\sum \frac{3L}{2}b_n$, and again by the comparison test, $\sum a_n$ converges. Q.E.D.

One of the applications of the limit comparison test is that it allows us to ignore small terms. Consider the series $\sum a_n = \sum \frac{3n^2+2n+1}{n^3+1}$. We can replace this series by $\sum b_n = \sum \frac{n^2}{n^3} = \sum \frac{1}{n}$ because

$$\frac{a_n}{b_n} = \frac{3n^2+2n+1}{n^3+1}\Big/\frac{n^2}{n^3} = \frac{3n^2+2n+1}{n^2}\,\frac{n^3}{n^3+1} \to 3 \cdot 1 = 3$$

But $\sum b_n$ is the harmonic series, which diverges. Therefore our original series $\sum a_n$ also diverges.

The root test. The root test doesn't have a lot of applications, but I'm including it here since it's one of the standard tests. For the root test, you look at the limit of the n^{th} root of the n^{th} term.

Theorem 9 (The root test)**.** If $\lim_{n\to\infty} a_n^{1/n} = L$, and if $L < 1$ then the series converges, but if $L > 1$ the series diverges.

For the root test, if $L = 1$, then the test is inconclusive, so you have to use some other test.

Example 10. The root test is especially useful when the n^{th} term already has a n^{th} power in it. Consider the series $\sum \dfrac{1}{(\ln n)^n}$. Here, $a_n^{1/n} = \left(\dfrac{1}{(\ln n)^n}\right)^{1/n} = \dfrac{1}{\ln n} \to 0$. So that series converges.

Here's the proof for the root test in the case that $L < 1$. The case $L > 1$ is analogous. We'll show $\sum a_n$ converges by comparing it to a larger convergent geometric series. Let r be a number between L and 1. Since $a_n^{1/n} \to L$, therefore for sufficiently large n, $a_n^{1/n} < r$, so $a_n < r^n$. But the geometric series $\sum r^n$ converges, so $\sum a_n$ also converges. Q.E.D.

The ratio test. We won't use the root test a lot, but the ratio test is very important, and we'll use a version of it soon on every power series we analyze. The statement of it is similar to that of the root test. For the ratio test, you look at the limit of the ratio $\dfrac{a_{n+1}}{a_n}$ of adjacent terms.

Theorem 11 (The ratio test)**.** If $\lim_{n\to\infty} \dfrac{a_{k+1}}{a_k} = L$, and if $L < 1$ then the series converges, but if $L > 1$ the series diverges.

For the ratio test, as it was for the root test, if $L = 1$, then the test is inconclusive, so you have to use some other test.

Example 12. Consider the series $\sum \dfrac{7^n}{n!}$. The n^{th} term is $a_n = \dfrac{7^n}{n!}$ so the next term is $a_{n+1} = \dfrac{7^{n+1}}{(n+1)!}$. For this ratio test, we'll examine the ratio $\dfrac{a_{n+1}}{a_n}$ and find its limit.

$$a_{n+1} \big/ a_n = \frac{7^{n+1}}{(n+1)!} \Big/ \frac{7^n}{n!} = \frac{7^{n+1}}{7^n} \, \frac{n!}{(n+1)!} = \frac{7}{n+1} \to 0.$$

Since the limit $L = 0$ is less than 1, this series converges.

The proof of the first case of the ratio test depends on comparing it to a larger convergent geometric series like we did for the root test. Again, we'll do just the case $L < 1$ here since the case $L > 1$ has an analogous proof.

Let r be a number between L and 1. Since $\dfrac{a_{n+1}}{a_n}$, therefore for sufficiently large n, say $n \geq N$, we have $\dfrac{a_{n+1}}{a_n} < r$, and so $a_{n+1} < a_n r$. A quick inductive argument shows that $a_n < a_N r^{n-N}$. That says the tail of our series is dominated by a convergent geometric series, $\displaystyle\sum_{n=N}^{\infty} a_n < \sum_{n=N}^{\infty} a_N r^{n-N}$. Therefore, by the comparison test, the original series also converges. Q.E.D.

What test to use? We've got several example series and several convergence tests. When you're looking at a positive series, what's the best way to determine whether it converges or diverges?

This is more of an art than a science, that is, sometimes you have to try several things in order to find the answer. Here are a few pointers you can use.

If you recogize it as a geometric series $\sum ar^n$, you know if it converges or not. It converges when the ratio r is in the interval $(-1, 1)$.

If it's a p-series $\sum \frac{1}{n^p}$, you know if it converges or not. It converges when $p > 1$.

If the terms don't approach 0, you know it diverges.

If you can dominate a known divergent series with the series, it diverges. If you know a convergent series that dominates it, it converges.

The limit comparison test can be used to simplify the problem of convergence because it allows you to ignore small terms.

If it's got factorials or powers, you can usually use the ratio test. The root test also works on many of these.

The integral test works on many series, but you'll have to evaluate or bound the integral to make the determination.

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