Power Series
Math 122 Calculus III
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**Introduction to power series.** One of the main purposes of our study of series is to understand power series. A power series is like a polynomial of infinite degree. For example,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

is a power series. We'll look at this one in a moment.

Power series have a lot of properties that polynomials have, and that makes them easy to work with. Also, they're general enough to represent lots of important functions like $e^x$, $\ln x$, $\sin x$, and $\cos x$.

Let's look at $1 + x + x^2 + \cdots + x^n + \cdots$. For a fixed value of $x$, this is just a geometric series, and we know that it sums to $\dfrac{1}{1-x}$ when $x \in (-1, 1)$. For values of $x$ outside the interval $(-1, 1)$, the series diverges. Thus,

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad \text{for } x \in (-1, 1).$$

The interval where a power series converges is called the *interval of convergence*.

Since we know the series for $\dfrac{1}{1-x}$, we can use it to find power series for some other functions. For instance, if we subsititute $-x$ for $x$ we get

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots \qquad \text{for } x \in (-1, 1).$$

If, instead, we substitute $2x$ for $x$ we get

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots \qquad \text{for } x \in (-\tfrac{1}{2}, \tfrac{1}{2}),$$

and $-x^2$ for $x$ gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots \qquad \text{for } x \in (-1, 1).$$

Two of the important properties of polynomials that are shared with power series is that they can be differentiated and integrated term by term. That is, if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

for $x$ in some interval, then, differentiating,

$$f'(x) = a_1 + 2a_2 x + \cdots + na_n x^{n-1} + \cdots$$

for $x$ in the same interval; also, integrating,

$$\int f(x)\,dx = a_0 x + a_1 \frac{x^2}{2} + \cdots + a_n \frac{x^{n+1}}{n+1} + \cdots + C$$

again for $x$ in that interval. Integration works not only for indefinite integrals (as just written), but also for definite integrals.

Thus, integrating the series for $\dfrac{1}{1+x}$ between 0 and $x$, we get

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \qquad \text{for } x \in (-1,1).$$

We get the power series for $\arctan x$ by integrating the series for $\dfrac{1}{1+x^2}$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{6} + \cdots \qquad \text{for } x \in (-1,1).$$

We'll need more theory to develop the following power series for the three important functions $e^x$, $\cos x$, and $\sin x$. We'll do that next time. It leads to the important results that for all $x$,

$$e^x \;=\; 1 \;+\; x \;+\; \frac{x^2}{2!} \;+\; \frac{x^3}{3!} \;+\; \frac{x^4}{4!} \;+\; \frac{x^5}{5!} \;+\; \cdots$$

$$\cos x \;=\; 1 \qquad\qquad -\; \frac{x^2}{2!} \qquad\qquad +\; \frac{x^4}{4!} \qquad\qquad -\; \cdots$$

$$\sin x \;=\; \qquad\quad x \qquad\qquad -\; \frac{x^3}{3!} \qquad\qquad +\; \frac{x^5}{5!} \qquad\qquad \cdots\;.$$

Note that the terms of the $\cos x$ series are the even terms of the $e^x$ series, but the $\cos x$ series alternates sign, and the terms of the $\sin x$ series are the odd terms of the $e^x$ series, and, again, the $\sin x$ series alternates sign.

**Power series centered at numbers other than 0.** The power series mentioned above can be translated to other numbers if we make a linear substitution.

For example, take the power series for $\ln(1+x)$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \qquad \text{for } x \in (-1,1)$$

and subsititute $u = 1 + x$, so $x = u - 1$. Then

$$\ln u = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(u-1)^n}{n} = (u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \cdots + (-1)^{n+1}\frac{(u-1)^n}{n} + \cdots \qquad \text{for } u \in (0,2).$$

This new series expresses $\ln u$ as a series of powers of $u - 1$. This power series is said to be centered at 1. Note that the interval of convergence is translated by 1.

For another example, take our first power series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad \text{for } x \in (-1,1)$$

and substitute $u = x - 1$, so $x = u + 1$. Then we find

$$\frac{-1}{u} = 1 + (u + 1) + (u + 1)^2 + \cdots + (u + 1)^n + \cdots \quad \text{for } u \in (-2, 0).$$

This series is centered at $-1$.

For the most part, we'll use power series centered at 0, but sometimes power series centered at other numbers are useful.

In summary, a power series centered at a number $a$ is of the form

$$\sum_{n=0}^{\infty} a_n(x - a)^n = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots .$$

We'll see next that power series centered at a number $a$ have intervals of convergence centered around $a$, that is, the series will converge if $x$ lies between $a - r$ and $a + r$ where $r$ is half the length of the interval. This number $r$ is called the *radius of convergence*.

**Finding the radius of convergence.** There's a version of the ratio test which will usually be able to tell us what the radius of convergence of a power series is. It doesn't work for all possible power series, but it does for all the important ones.

Treat $x$ as a constant and apply the usual ratio test in conjuction with the absolute convergence test on a power series $\sum_{n=0}^{\infty} a_n(x - a)^n$ centered at $a$. Suppose that the absolute value of the ratio of the next term to the present term has a limit $L$.

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}(x - a)^{n+1}}{a_n(x - a)^n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x - a| = |x - a| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Let $r$ be the reciprocal of $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then $L = |x - a|/r$.

Now, if $L < 1$, the series absolutely converges, but if $L > 1$ it diverges. In terms of $r$, that says $|x - a| < r$, then the series absolutely converges, but if $|x - a| > r$ it diverges by the term test. In other words the power series converges in an interval centered at $a$ with radius $r$. Thus, we've proven the following theorem.

**Theorem 1** (Abel)**.** If the ratio $|a_{n+1}/a_n|$ of the absolute values of the coefficients of a power series has a limit, then the reciprocal of that limit is the radius of convergence of the power series. More precisely, if the series is $\sum_{n=0}^{\infty} a_n(x - a)^n$, and $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{r}$, then the series absolutely converges for $x \in (a - r, a + r)$, and the series diverges for $|x - a| > r$.

Note that this theorem says nothing about convergence at the endpoints of the interval of convergence. Those two values, $x = a \pm r$, need to be checked separately for convergence.

**Example 2.** Consider the power series $\sum_{n=1}^{\infty} \frac{1}{n2^n}(x - 3)^n$.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{(n + 1)2^{n+1}} \Big/ \frac{1}{n2^n} = \frac{n}{2(n + 1)} \to \frac{1}{2}$$

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So the radius of convergence is 2, and the interval of convergence goes from 1 to 5.

Now let's check the two endpoints for convergence. At $x = 5$ the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, which we know diverges. At $x = 2$ the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series, which we know converges. Thus, the interval of convergence for this series is the half open interval $[2, 5)$.

**Extension of the theory to complex numbers.** Everything mentioned so far applies only to real numbers, but it can all be extended to complex numbers. In particular, consider the last theorem. Let a complex power series be $\sum_{n=0}^{\infty} a_n(z - a)^n$, where $a$ and each $a_n$ is a complex number and $z$ is a complex variable. If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{r}$, then the series converges when $|z - a| < r$, that is, it converges in a circle centered at the point $a$ with radius $r$. (Note that the limit is the limit of real numbers and $r$ is a real number.)

For example, the series in the last example converges inside the circle of radius 2 centered at 3. It will diverge outside that circle. It will converge for some points on the circle and diverge for others.

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