Math 130 Linear Algebra. Selected answers from section 3.1.

2. Determine whether each of the following permutations is even or odd.

a. 4213. There are four inversions: 42, 41, 43, and 21, so it's even.

b. 1243. There is one inversion: 43, so it's odd.

3. Determine the sign associated with the following permutations.

a. 25431. The inversions are 21, 54, 53, 51, 43, 41, 31. There are 7 of them, so the permutation is odd, and the sign is negative.

b. 31245. The inversions are 31, 32. There are 2 of them, so the permutation is even, and the sign is positive.

4. In each of the following pairs of permutations verify that the number of inversions differs by an odd number.

They're supposed to differ by an odd number because the second permutation is obtained by interchanging two numbers in the first permutation. The purpose of this exercise is to illustrate that general theorem.

a. 436215 and 416235.

Inversions in 436215: 43, 42, 41, 32, 31, 62, 61, 65, 21.
Inversions in 416235: 41, 42, 43, 62, 63, 65.
Since $9-6=3$ is odd, the number of inversions differs by an odd number.

b. 632415 and 523416.

Inversions in 623415: 62, 63, 64, 61, 65, 21, 31, 41.
Inversions in 523416: 52, 53, 54, 51, 21, 31, 41.
Since $8-7=1$ is odd, the number of inversions differs by an odd number.

6. Evaluate the determinants by the definition. The definition of the determinant of an $n \times n$ matrix has $n!$ terms, each term being a product of n elements of the matrix, one chosen from each row and column. When $n=2$, this is the best way to compute the determinant. When $n=3$, it's not bad, but when $n>3$, there are so many terms to evaluate that the definition is a lot of work, except when the matrix has a lot of zeros in it.

a.

$$\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = 2 \cdot 3 - 1 \cdot 4 = 2$$

b.

$$\begin{aligned} \begin{vmatrix} 0 & 0 & -2 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{vmatrix} &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + \\ & \quad a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &= -(-2) \cdot 3 \cdot 4 = 24 \end{aligned}$$

15. Compute the indicated determinants.

Since all the matrices in exercises 15 and 16 are diagonal or triangular, their determinants are computed by taking the product of the entries on the diagonals.

20. Evaluate the determinant by reducing the matrix to triangular form.

a. There are many ways to reduce this matrix to triangular form, but they all lead to the same value for the determinant. Here's one where you make the matrix into lower triangular form.

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} &= -\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \\ &= -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \\ &= -1 \end{aligned}$$

b. The matrix is already in lower triangular form, so the determinant is the product of the elements on the diagonal. $2 \cdot 3 \cdot 4 \cdot (-5) = -120$.