Math 130 Linear Algebra. More selected answers from section 3.1.

T.1. Show that if we interchange two numbers in the permutation $j_1 j_2 \ldots j_n$, then the number of inversions is either increased or decreased by an odd number. (Hint: First show that if two adjacent numbers are interchanged, then the number of inversions is either increased or decreased by 1. Then show that an interchange of any two numbers can be achieved by an odd number of successive interchanges of adjacent numbers.)

Let's follow the hint. Suppose two adjacent numbers, j_i and j_{i+1}, in the permutation $j_1 j_2 \ldots j_i j_{i+1} \ldots j_n$ are interchanged. Then the new permutation will be $j_1 j_2 \ldots j_{i+1} j_i \ldots j_n$. An element j_r will be in the same position relative to any other element j_s with the one exception of the two adjacent elements, j_i and j_{i+1}, which are exchanged. Thus, the total number of inversions will be the same as before except it will go down by one if $j_i > j_{i+1}$, or go up by one if $j_i < j_{i+1}$. Thus, in this special case, the number of inversions is either increased or decreased by an odd number.

Now let's consider the general interchange where a number j_i is interchanged with j_{i+p}, which is p numbers to the right of j_i. First successively interchange j_i with j_{i+1}, then j_{i+2}, up through j_{i+p}. That's p adjacent interchanges. Next successively interchange j_{i+p} with j_{i+p-1}, then j_{i+p-2}, down through j_{i+1}. That's $p-1$ interchanges, making a total of $2p-1$, an odd number, of interchanges. The resulting permutation is the same as the original permutation $j_1 j_2 \ldots j_n$, except the two numbers j_i and j_{i+p} are interchanged. To achieve this interchange, an odd number of adjacent interchanges was performed, each increasing or decreasing the number of inversions by one, therefore the total number of inversions is either increased or decreased by an odd number. Q.E.D.

T.6. Is $|AB| = |BA|$? Justify your answer.

(Although it doesn't say so, it is assumed that A and B are both square matrices. If they're not, the statement $|AB| = |BA|$ can still make sense if A is an $m \times n$ matrix and B is an $m \times n$ matrix, for then both products AB and BA are square matrices, so they both have determinants. It turns out it's not true in that case.)

It is enough to give one counterexample to show something is false, but you need a general argument to show something is true. In this case, the statement is true, so it is not enough to choose one matrix A and one matrix B and check that $|AB| = |BA|$. That is to say, an example is not enough to justify a general statement.

One way to prove this statement, at least when $n = 2$, is to choose generic matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then compute the products AB and BA, then their determinants, and show you get the same thing.

But there's a far easier way. Since $|AB| = |A|\,|B|$ and $|BA| = |B|\,|A|$, and, for numbers, multiplication is commutative, therefore $|AB| = |A|\,|B| = |B|\,|A| = |BA|$.

T.12. Evaluate the Vandermonde determinant as given.

First, expand the determinant:

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = a^2b + ac^2 + b^2c - bc^2 - ab^2 - a^2c$$

Next, expand what it's supposed to equal:

$$\begin{aligned} & (b-a)(c-a)(b-c) \\ = \; & b(c-a)(b-c) - a(c-a)(b-c) \\ = \; & bc(b-c) - ab(b-c) - ac(b-c) + a^2(b-c) \\ = \; & b^2c - bc^2 - ab^2 + abc - abc + ac^2 + a^2b - a^2c \\ = \; & b^2c - bc^2 - ab^2 + ac^2 + a^2b - a^2c \end{aligned}$$

Except for the order of the terms, these are the same.

What's interesting about the Vandermonde determinant $V(a, b, c)$ is that it will tell you if any of the numbers a, b, and c are equal since that happens exactly when $V(a, b, c) = 0$.

Of course, there are also Vandermonde determinants for more than three symbols.