Math 130 Linear Algebra. Selected answers 3.2.

4.a. Evaluate the determinant

$$|A| = \begin{vmatrix} 2 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 4 & 1 \\ 2 & 3 & 0 & 0 \end{vmatrix}$$

by cofactor expansion along your chosen row or column.

Since there are two 0s in the last row, it makes sense to expand along it. You'll get -2 times one 3×3 minor, plus 3 times another 3×3 minor, minus 0 times something, plus 0 times something.

$$\begin{aligned} |A| &= -2\begin{vmatrix} 2 & -3 & 1 \\ 1 & 2 & -1 \\ -1 & 4 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & -3 & 1 \\ 0 & 2 & -1 \\ 3 & 4 & 1 \end{vmatrix} - 0 + 0 \\ &= -2 \cdot 18 + 3 \cdot 15 \\ &= 9 \end{aligned}$$

10. Compute the inverses of the given matrices, if they exist, using the formula $A^{-1} = \dfrac{\text{adj } A}{|A|}$.

a. The matrix is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -3 & 4 \end{bmatrix}$$

The adjoint matrix is the transpose of the the cofactors, so it is

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 3 & 3 \end{bmatrix}$$

and if you divide the adjoint matrix by the determinant, which is $|A| = ad - bc = 18$, you get the inverse matrix

$$\begin{aligned} A^{-1} = \frac{\text{adj } A}{|A|} &= \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \tfrac{1}{18}\begin{bmatrix} 4 & -2 \\ 3 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4/18 & -2/18 \\ 3/18 & 3/18 \end{bmatrix} \\ &= \begin{bmatrix} 2/9 & -1/9 \\ 1/6 & 1/6 \end{bmatrix} \end{aligned}$$

Thus, the adjoint formula for the inverse matrix, namely, $A^{-1} = \dfrac{\text{adj } A}{|A|}$, generalizes the formula we saw a long time ago for finding the inverse of a 2×2 matrix

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{bmatrix}$$

where Δ is the determinant of A.

b. The matrix is

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

The matrix of minors is

$$\begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & \begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & \begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

which is

$$\begin{bmatrix} 3 & -2 & -1 \\ 6 & 10 & -2 \\ 2 & 8 & 4 \end{bmatrix}$$

To get the adjoint matrix, negate every other term in this matrix in a checkerboard fashion, and transpose the matrix.

$$\text{adj } A = \begin{bmatrix} 3 & -6 & 2 \\ 2 & 10 & -8 \\ -1 & 2 & 4 \end{bmatrix}$$

Now divide every element of the adjoint matrix by the determinant $|A|$. When you compute the determinant of A, you find that $|A| = 14$. Therefore, the inverse matrix is

$$A^{-1} = \begin{bmatrix} 3/14 & -6/14 & 2/14 \\ 2/14 & 10/14 & -8/14 \\ -1/14 & 2/14 & 4/14 \end{bmatrix}$$

20. Use Cramer's rule to solve the system of equations

$$\begin{array}{rcrcrcr} 2x & + & 4y & + & 6z & = & 2 \\ x & & & + & 2z & = & 0 \\ 2x & + & 3y & - & z & = & -5 \end{array}$$

First, compute the determinant Δ of the 3×3 coefficient matrix.

$$\Delta = \begin{vmatrix} 2 & 4 & 6 \\ 1 & 0 & 2 \\ 2 & 3 & -1 \end{vmatrix} = 26$$

Next, replace the first column by the constant vector, and compute that determinant.

$$\Delta_x = \begin{vmatrix} 2 & 4 & 6 \\ 0 & 0 & 2 \\ -5 & 3 & -1 \end{vmatrix} = -52$$

Then in the unique solution, $x = \Delta_x/\Delta = -2$. Next, replace the second column by the constant vector, and compute that determinant.

$$\Delta_y = \begin{vmatrix} 2 & 2 & 6 \\ 1 & 0 & 2 \\ 2 & -5 & -1 \end{vmatrix} = 0$$

So $y = \Delta_y/\Delta = 0$. Likewise, replace the third column by the constant vector.

$$\Delta_z = \begin{vmatrix} 2 & 4 & 2 \\ 1 & 0 & 0 \\ 2 & 3 & -5 \end{vmatrix} = 26$$

which gives $z = 1$. Thus, the unique solution is $(x, y, z) = (-2, 0, 1)$.