Math 130 Linear Algebra. More selected answers from section 4.2.

25. If possible, find a, b, and c not all zero so that $\mathbf{v} = (a, b, c)$ is orthogonal to both

$$\mathbf{w} = (1, 2, 1) \text{ and } \mathbf{x} = (1, -1, 1).$$

We need both inner products $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{v} \cdot \mathbf{x}$ to be 0. That gives us the system of linear equations

$$\begin{cases} a + 2b + c = 0 \\ a - b + c = 0 \end{cases}$$

The general solution to this system is $(a, b, c) = (-c, 0, c)$ where c is arbitrary. Thus, one answer is $(a, b, c) = (-1, 0, 1)$.

26. Verify the triangle inequality for $\mathbf{u} = (1, 2, 3, -1)$ and $\mathbf{v} = (1, 0, -2, 3)$.

We need to show that

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

That's the inequality

$$\|(2, 2, 1, 2)\| \leq \|(1, 2, 3, -1)\| + \|(1, 0, -2, 3)\|,$$

which simplifies to

$$\sqrt{13} \leq \sqrt{15} + \sqrt{14},$$

and that's obviously true.

31. Verify that the triangle with vertices $P_1 = (2, 3, -4)$, $P_2 = (3, 1, 2)$, and $P_3 = (7, 0, 1)$ is isoceles.

We don't know which two sides are going to have the same length, so let's find the lengths of all three sides

$$\begin{aligned} \|P_2 - P_1\| &= \|(1, 2, 6)\| = \sqrt{41} \\ \|P_3 - P_2\| &= \|(-6, -1, 2)\| = \sqrt{41} \\ \|P_1 - P_3\| &= \|(5, 3, -8)\| = \sqrt{98} \end{aligned}$$

Aha! Two sides have the same length.

32. Verify that the triangle with vertices $P_1 = (2, 3, -4)$, $P_2 = (3, 1, 2)$, and $P_3 = (7, 0, 1)$ is a right triangle.

One way is to show that one of the three angles is a right angle. It could occur at any of the three vertices, and each needs to be checked. Let's check to see if the right angle occurs at P_2. Let $\mathbf{u}$ and $\mathbf{v}$ be the displacement vectors representing the two sides of the triangle adjacent to P_2.

$$\begin{aligned} \mathbf{u} &= P_1 - P_2 = (-1, 2, -6) \\ \mathbf{v} &= P_3 - P_2 = (4, -1, -1) \end{aligned}$$

Are these $\perp$? Since $\mathbf{u} \cdot \mathbf{v} = -4 - 2 + 6 = 0$, therefore they're $\perp$, and the right angle occurs at P_2.

An alternate method is to compute the squares of the lengths of the three sides and show that one of the three is the sum of the other two.

34. The vector $\mathbf{u} = (20, 30, 80, 10)$ gives the number of receivers, CD players, speakers, and cassette recorders [what are those things, anyway?] that are on hand at a stereo shop. [I wonder if the shop has any old Victrolas or Gramophones.] The vector $\mathbf{v} = (200, 120, 80, 70)$ gives the price (in dollars) of each receiver, CD player, speaker, and cassette recorder, respectively. What does the dot product $\mathbf{u} \cdot \mathbf{v}$ tell the shop owner?

It's the sum of the prices of all the receivers, CD players, etc. In other words, it's the worth of the shop inventory of those items.

T.7. Show that

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.$$

Proof. Let $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{v} = (v_1, \ldots, v_n)$, and $\mathbf{w} = (w_1, \ldots, w_n)$. Then the LHS equals

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= (u_1, \ldots, u_n) \cdot (v_1 + w_1, \ldots, v_n + w_n) \\ &= u_1(v_1 + w_1) + \cdots + u_n(v_n + w_n) \\ &= u_1v_1 + u_1w_1 + \cdots + u_nv_n + u_nw_n \end{aligned}$$

while the RHS equals

$$\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = (u_1v_1 + \cdots + u_nv_n) + (u_1w_1 + \cdots + u_nw_n).$$

These are clearly equal. (Note: we used distributivity of multiplication over addition, associativity of addition, and commutativity of addition for scalars.) Q.E.D.

T.10. (Pythagorean Theorem in $\mathbf{R}^2$.) Show that

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

if and only if

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

Proof. Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$. Then the first equation says

$$(u_1 + v_1)^2 + \cdots (u_n + v_n)^2 = (u_1^2 + \cdots + u_n^2) + (v_1^2 + \cdots + v_n^2).$$

Subtracting the RHS from the left, we see that's equivalent to

$$2u_1v_1 + \cdots 2u_nv_n = 0.$$

Dividing by 2, we see that condition, in turn, is equivalent to $\mathbf{u} \cdot \mathbf{v} = 0$. Q.E.D.