Math 130 Linear Algebra. Selected answers from section 4.3

4. Which of the following are linear transformations?

a. $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_1^2 + u_2 \\ u_1 - u_3 \end{bmatrix}$

That square u_1^2 is the giveaway that this L is not linear. To show explicitly that it's not linear you could show that $L(c\mathbf{u}) \neq cL(\mathbf{u})$.

$$\begin{aligned} L(c\mathbf{u}) &= L\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \\ cu_4 \end{bmatrix}\right) \\ &= \begin{bmatrix} cu_1 \\ c^2u_1^2 + cu_2 \\ cu_1 - cu_3 \end{bmatrix} \\ &\neq \begin{bmatrix} cu_1 \\ cu_1^2 + cu_2 \\ cu_1 - cu_3 \end{bmatrix} = cL(\mathbf{u}) \end{aligned}$$

b. $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Yes, this is a linear transformation because all matrix transformations are linear transformations.

c. $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Yes, this is a linear transformation. The two required equations $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$ and $L(k\mathbf{u}) = kL(\mathbf{u})$ are both satisfied since both their sides equal $\mathbf{0}$.

21. Describe the following linear transformations geometrically.

a. $L(x, y) = (-x, y)$. Since a point (x, y) is sent to the point $(-x, y)$, it is moved to the other side of the y-axis, and the same distance away from the y-axis, but its height doesn't change. In other words, the point is reflected across the y-axis. You can describe the transformation as a reflection across the y-axis.

b. $L(x, y) = (-x, -y)$. Both coordinates are negated this time. You could describe it as being the composition of two reflections, one across the y-axis, the other across the x-axis. Another way to look at it is as an 180° rotation around the origin. So, this is a *half-turn* around the origin. Yet you could also say that a point (x, y) is sent through the origin $(0, 0)$ to the point $(-x, -y)$ on the other side of the origin. So this is a *point inversion* through the origin. Any of these three descriptions is just fine.

c. $L(x, y) = (-y, x)$. This is not so easy as the first two since the coordinates are being exchanged as well as one being negated. Sometimes it's easiest to see the transformation by examining what happens to the standard unit vectors.

$$L(\mathbf{i}) = L(1, 0) = (0, 1) = \mathbf{j}$$

$$L(\mathbf{j}) = L(0, 1) = (-1, 0) = -\mathbf{i}$$

Keep in mind that every linear transformation fixes the origin. What's happinging then, is that $\mathbf{i}$ is sent to $\mathbf{j}$, and $\mathbf{j}$ is sent to $-\mathbf{i}$, so we have a rotation of 90° counterclockwise around the origin. (We've seen this one before a couple of times.)

22. Describe the following linear transforamtions geometrically.

a. $L(x, y) = (y, x)$. The standard basis vectors $\mathbf{i}$ and $\mathbf{j}$ are exchanged, while the origin is fixed, so this is a reflection across a diagonal line through the origin. You'd say it's a reflection with $y = x$ being the axis of reflection.

b. $L(x, y) = (-y, -x)$. This sends $\mathbf{i}$ to $-\mathbf{j}$, and it sends $\mathbf{j}$ to $-\mathbf{i}$. It's a little harder to see, but it's also a reflection across a diagonal line through the origin, but this time the axis of reflection is the other diagonal line $y = -x$.

c. $L(x, y) = (2y, 2x)$. Every point is sent twice as far away from the origin. You can call that a scaling, or an expansion, or a dilation, by a factor of 2.