Math 130 Linear Algebra. More selected answers from section 4.3

26. Represent this linear transformation with a matrix.

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$$

Just fill in the 2×2 matrix so the multiplication works

$$\begin{bmatrix} * & * \\ * & * \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$$

You'll see that the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

does it.

That's equivalent to seeing where the standard basis elements go and then placing the images in the columns in the matrix. For example $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is sent to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so that's the first column of the matrix.

27. Represent a counterclockwise rotation through $\frac{\pi}{3}$ radians.

In general, a counterclockwise rotation through an angle θ is given by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

When $\theta = \frac{\pi}{3} = 60°$, you know $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$, so the matrix looks like

$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

29. Represent this linear transformation with a matrix.

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+z \\ y+z \end{bmatrix}$$

Fill in the matrix as described in exercise 26 above.

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y \\ x+z \\ y+z \end{bmatrix}$$

T9. Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation defined by $L(\mathbf{u}) = A\mathbf{u}$ where

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

For $\phi = 30°$, L defines a counterclockwise rotation by $30°$.

a. If $T_1(\mathbf{u}) = A^2\mathbf{u}$, describe the action of T_1 on $\mathbf{u}$.

Assuming that $\phi = 30°$ throughout this exercise, then T_1 is the result of applying L twice, that is, it rotates $\mathbf{u}$ counterclockwise by $2\phi = 60°$.

b. If $T_2(\mathbf{u}) = A^{-1}\mathbf{u}$, describe the action of T_2 on $\mathbf{u}$.

T_2 is the result of applying the inverse of L, that is, it rotates $\mathbf{u}$ counterclockwise by $-\phi = -30°$, in other words, it rotates $\mathbf{u}$ clockwise $30°$.

c. What is the smallest positive value of k for which $T(\mathbf{u}) = A^k\mathbf{u} = \mathbf{u}$?

In other words, how many times to you have to apply this $30°$ rotation to move a vector back where it started? Since $360°/30°$ equals 12, the answer is $k = 12$.

T10. Let $O : \mathbf{R}^n \to \mathbf{R}^n$ be the zero linear transformation defined by $O(\mathbf{v}) = \mathbf{0}$ for $\mathbf{v} \in \mathbf{R}^n$. Find the standard matrix representing O.

It's clearly a square $n \times n$ matrix filled with zeros, that is, it's the zero matrix $\mathbf{0}$.

T11. Let $I : \mathbf{R}^n \to \mathbf{R}^n$ be the identity linear transformation defined by $I(\mathbf{v}) = \mathbf{v}$ for $\mathbf{v} \in \mathbf{R}^n$. Find the standard matrix representing I.

It's clearly the identity matrix.