Math 130 Linear Algebra. Selected answers from section 5.1

2. Compute $\mathbf{u}\times\mathbf{v}$.

a. $\mathbf{u} = (1,-1,2)$, $\mathbf{v} = (3,1,2)$.

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 2\\ 3 & 1 & 2\end{vmatrix}\\ &= -4\mathbf{i} - (-4)\mathbf{j} + 4\mathbf{k}\\ &= (-4,4,4)\end{aligned}$$

b. $\mathbf{u} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$, $\mathbf{v} = \mathbf{i}+3\mathbf{k}$.

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -2\\ 1 & 0 & 3\end{vmatrix}\\ &= 3\mathbf{i} - 8\mathbf{j} - \mathbf{k}\\ &= (3,-8,-1)\end{aligned}$$

9. Find the area of the triangle with vertices $P_1 = (1,-2,3)$, $P_2 = (-3,1,4)$, and $P_3 = (0,4,3)$.

The area of the triangle is half the length of the cross product of any two vectors representing the sides of the triangle. Choose any one of the three vertices as the base point, say P_1, then the displacements from that base point to the other two vertices are the vectors. So $\mathbf{u} = P_2 - P_1 = (-4,3,1)$, and $\mathbf{v} = P_3 - P_1 = (-1,6,0)$. Therefore,

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -4 & 3 & 1\\ -1 & 6 & 0\end{vmatrix} = (-6,-1,-21).$$

The area of the triangle is

$$\tfrac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \tfrac{1}{2}\sqrt{(-6)^2+(-1)^2+(-21)^2} = \tfrac{1}{2}\sqrt{478}.$$

10. Find the area of the triangle with vertices P_1, P_2, and P_3 where one side is $P_1P_2 = 2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and another side is $P_1P_3 = 3\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

We've got two vectors representing the sides of the triangle, namely $\mathbf{u} = (2,3,-1)$, and $\mathbf{v} = (1,2,2)$. Therefore,

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 3 & -1\\ 1 & 2 & 2\end{vmatrix} = (8,5,1).$$

The area of the triangle is

$$\tfrac{1}{2}\sqrt{64+25+1} = \tfrac{1}{2}\sqrt{90}.$$

11. Find the area of the parallelogram with adjacent sides $\mathbf{u} = \mathbf{i}+3\mathbf{j}-2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i}-\mathbf{j}-\mathbf{k}$.

All we have to do is find the length of the cross product $\mathbf{u}\times\mathbf{v}$. The cross product is

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 3 & -2\\ 3 & -1 & -1\end{vmatrix} = (-5,5,-10).$$

Its length is

$$\|\mathbf{u}\times\mathbf{v}\| = \|(-5,5,-10)\| = \sqrt{150}.$$

12. Find the volume of the parallelepiped with a vertex at the origin and edges $\mathbf{u} = 2\mathbf{i}-\mathbf{j}$, $\mathbf{v} = \mathbf{i}-2\mathbf{j}-2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i}-\mathbf{j}+\mathbf{k}$.

The absolute value of the triple scalar product

$$[\mathbf{u},\mathbf{v},\mathbf{w}] = \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$$

is the volume of the parallelepiped with edges $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. In this case that determinant is

$$\begin{vmatrix}2 & -1 & 0\\ 1 & -2 & -2\\ 3 & -1 & 1\end{vmatrix} = -4+6+1-4 = -1$$

Therefore, the volume is $|-1| = 1$.

T.2. Show that $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$.

Since $\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$, therefore $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = u_2v_3w_1 - u_3v_2w_1 + u_3v_1w_2 - u_1v_3w_2 + u_1v_2w_3 - u_2v_1w_3$.

Also, $\mathbf{v}\times\mathbf{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)$, therefore $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1$.

The same six terms are in each expression, so they're equal.

T.4. Show that $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$.

The expansion for $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ is above in T.2. Since $\begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix} = u_1v_2w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1 - u_2v_1w_3 - u_1v_3w_2$, we have the same six terms, so they're equal.

T.5. Show that $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.

You could give either an algebraic or a geometric argument for this. For an algebraic argument, you would start by noting that $\mathbf{u}$ and $\mathbf{v}$ are parallel if an only if there is a nonzero scalar c such that $c\mathbf{u} = \mathbf{v}$, and work from there.

Here's a geometric argument. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ means that the parallelgram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ has area 0. That happens exactly when it's a degenerate parallelogram, that is, when $\mathbf{u}$ and $\mathbf{v}$ lie in a line, that is, they're parallel vectors.

T.7. Prove the *Jacobi identity*:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}.$$

Note: Although the cross product is not associative, it does satisfy this identity. It's a fairly strong identity since it's valid for any three vectors.

Proof. You can prove this identity in many different ways, but the easiest relies on one of the identities mentioned in the text, namely, the first listed below. Permuting the three vectors, we get two other two identities

$$\begin{array}{rcl}(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} & = & (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \\ (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} & = & (\mathbf{v} \cdot \mathbf{u})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \\ (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} & = & (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v}\end{array}$$

If we add the three equations together, then the LHS becomes the LHS of the Jacobi identity, while the RHS cancels to $\mathbf{0}$, since the dot product does commute. Q.E.D.