Math 130 Linear Algebra. Selected answers from section 6.1.

11. Determine whether the set of all ordered triples of real numbers (x, y, z) with the operations

$$(x, y, z) \oplus (x', y', z') = (x', y + y', z')$$

and

$$c \odot (x, y, z) = (cx, cy, cz)$$

is a vector space.

For addition, the middle coordinate looks like it ought to be okay, but the first and last look flaky. Addition is supposed to be commutative, but does

$$(x, y, z) \oplus (x', y', z') = (x', y', z') \oplus (x, y, z)$$

for all values of x, y, and z? That requires $(x', y + y', z')$ to be equal to $(x, y + y', z)$ for all values of x, y, and z. But that's not true for many values. For instance, when $x = 0$ and $x' = 1$ it's false.

You can find other reasons it's not a vector space. For instance, it does not have an additive identity $\mathbf{0}$. The operation $\oplus$ is, however, associative.

20a. If V is a vector space that has a nonzero vector, then how many vectors are in V?

Since every scalar multiple of the nonzero vector lies in V, and all the scalar multiples are distinct, therefore there are infinitely many vectors in V.

20b. Describe all vector spaces having a finite number of vectors.

If the space doesn't have infinitely many vectors, then it has no nonzero vector. Thus, it is just the trivial vector space $\mathbf{0}$.

T3. Show that cancellation holds in a vector space, that is, if $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

You can give your proof in varying degrees of completeness. Let's prove it carefully using only the axioms of vector spaces.

Suppose that

$$\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}.$$

Add $(-\mathbf{u})$ to each side of the equation to get

$$(-\mathbf{u}) + (\mathbf{u} + \mathbf{v}) = (-\mathbf{u}) + (\mathbf{u} + \mathbf{w}).$$

Now, by associativity of vector addition

$$(-\mathbf{u} + \mathbf{u}) + \mathbf{v} = (-\mathbf{u} + \mathbf{u}) + \mathbf{w}.$$

But $-\mathbf{u} + \mathbf{u} = \mathbf{0}$, so

$$\mathbf{0} + \mathbf{v} = \mathbf{0} + \mathbf{w}.$$

But $\mathbf{0}$ is the additive identity. Therefore, $\mathbf{v} = \mathbf{w}$.

Thus, we have shown that $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$ implies $\mathbf{v} = \mathbf{w}$.

Note that you cannot use coordinates for this proof, since a coordinate proof will only prove the statement for vector spaces of the form $\mathbf{R}^n$, not for all vector spaces. Not all vector spaces have coordinates.