Math 130 Linear Algebra

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Wednesday, 2 Sep 2009, second class

Due Today. Exercises from section 1.1, page 8, numbers 1–4, 13–14, 21–22, 23, and T4.

Read for Friday. Finish 1.2 on matrices and read 1.3 on dot products and matrix multiplication.

Due Friday. Exercises from section 1.2: 1–2, 4–7 parts a–d each, 8, 9, T1, T5a.

No class Monday. Martin Luther King Jr. Day.

Due Wednesday. Exercises from section 1.3: 1–4, 7, 9, 11–12, 19–20, 33, T1, T4, ML1, ML2, ML5, ML6ac.

The ML exercises are MATLAB exercises. Do them, but you don't have to make printed versions of them. Feel free to use MATLAB to help you with your homework, even the parts of MATLAB we haven't discussed in class.

Last time. We discussed systems of linear equations. Sometimes they've got exactly one solution, sometimes no solutions, and sometimes infinitely many solutions. We saw the method of elimination as the ancient Chinese did it, how they created a matrix from the system of linear equations, first put the matrix into echelon form, then into reduced echelon form, and then they could read the solution right from the resulting matrix.

We'll formalize this method soon.

Today. We'll review some of the topics in section 1.1 about linear systems, and we'll begin discussing section 1.2 which introduces matrices.

Standard notation and terminology. When we've got a system of linear equations, we'll usually let m denote the number of equations, and n the number of variables, and say the system is a $m \times n$ system of equations. For instance, here's an example 3×2 system of equations:

$$\left\{ \begin{array}{rcrcr} 5x & + & 2y & = & 12 \\ 3x & - & y & = & 5 \\ x & + & 3y & = & 5 \end{array} \right.$$

Here, $m = 3$ since there are 3 equations, and $n = 2$ since the two variables are x and y. (Typically, when a system has more equations than unknowns, then there are no solutions. Does this system have any solutions?)

When there are many variables, we usually subscript them, but when there are few variables we ususally use different letters for the variables. For instance, if $n = 6$, we might use the 6 variables x_1, x_2, x_3, x_4, x_5, and x_6, but if $n = 3$, we might use the three variables x, y, and z.

A general $m \times n$ system looks like this:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

where the n variables are $x_1, x_2, \ldots, x_n$. The m constants that appear on the right side of the m equations are denoted $b_1, b_2, \ldots, b_m$. Since each of the m equations has n coefficients for the n variables, there are mn coefficients in all, and these are denoted with doubly subscripts, so, for instance a_{35}

is the coefficient of the variable x_5 in the third equation.

A *solution* to the system is an n-tuple of values for the n variables that make all the equations simultaneously true. For example, in the 3×2 system above, a solution is $(x, y) = (2, 1)$. In fact, it's the only solution for that system.

Matrices. A matrix is a rectangular array of numbers. You can see how they're related to systems of equations since the system is determined by its coefficients and its constants, and those can naturally be displayed in a matrix. Consider the 3×2 system of equations above. Its coefficients can be displayed in the 3×2 matrix

$$\begin{bmatrix} 5 & 2 \\ 3 & -1 \\ 1 & 3 \end{bmatrix}$$

called the *coefficient matrix* for the system. If you want to include the constants a a matrix, too, you can add another column to get the 3×3 matrix

$$\begin{bmatrix} 5 & 2 & 12 \\ 3 & -1 & 12 \\ 1 & 3 & 5 \end{bmatrix}$$

called the *augmented matrix* for the system. Frequently, a vertical line is added to separate the coefficients from the constants, as in

$$\left[\begin{array}{cc|c} 5 & 2 & 12 \\ 3 & -1 & 12 \\ 1 & 3 & 5 \end{array}\right]$$

Standard terminology and notation for matrices. When a single symbol is used to denote an entire matrix, it's usually a capital letter, like A or B.

Usually m is used for the number of rows and n for the number of columns. Such a matrix is called an $m \times n$ matrix. When $m = n$ we say the matrix is a *square matrix.* When symbols are used for the elements (that is, entries) of a matrix, they are often doubly indexed (that is, subscripted), and the indices indicate where the entry is located. For instance, a_{34} indicates the element in the 3rd row and 4th column. Note that the first index gives the row number and the second index gives the column number. When a generic row is needed, usually i is used, and when a generic column is needed, usually j is used. So a_{ij} is the element in the ith row and jth column.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Special square matrices. Square matrices were mentioned above. There are some special square matrices we'll discuss in class including diagonal matrices, scalar matrices, and the identity matrix. A square matrix for which $a_{ij} = 0$ whenever $i \neq j$ is called a *diagonal matrix.* The *main diagonal* of a square matrix consists of the elements a_{ii}. The main diagonal is usually just called the *diagonal.* Thus, a diagonal matrix only has nonzero entries on its diagonal. If all the entries of a diagonal matrix are the same, it's called a *scalar matrix.* The most important scalar matrix has all 1's on its diagonal, and that matrix is called an *identity matrix*, and it's usually denoted I. Thus, an identity matrix has 1's on its diagonal and 0's elsewhere.

Some of the homework exercises talk about upper-triangular matrices and lower-triangular matrices.

Row and column vectors. If a matrix has only one row, that is, if $m = 1$, then we'll call it a *row vector.* Likewise, if it has only one column, that is, if $n = 1$, then we'll call it a *column vector.*

Equality of matrices. Two matrices are equal when they have the same shape and all their corresponding entries are equal.

Addition and subtraction of matrices. If two matrices are the same shape, then you can add them together to produce another matrix by adding the corresponding elements together. Likewise you can subtract one matrix from another if they have the same shape. Examples 11 and 13 in the text illustrate when these operations are useful.

Scalar multiplication. The word "scalar" in this subject means "number," and by that we usually mean real number. Later in the course we'll take our field of scalars to be the field of complex numbers or some other field. For now we'll stick to reals. We'll use the standard notation **R** for the set of all real numbers (that is, all positive numbers, negative numbers, and 0).

Scalar multiplication is an operation where you can multiply a scalar and a matrix together. It's done by multiplying every entry in the matrix by that scalar. For instance, if A is a matrix, then $2A$ is found by doubling every entry in A. Note that $2A = A + A$, just as you would expect. Also, when A and B are two matrices, then $A-B = A+(-1)B$, just as you would expect.

Linear combinations. In mathematics, you say that a linear combination of things is a sum of multiples of those things. So, for example, one linear combination of the matrices A, B, and C is $2A+3B-4C$. In other words, a linear combination of matrices is found by the operations of addition, subtraction, and scalar multiplication.

The transpose of a matrix. One more simple operation we can discuss now is *transposition*. It's where you exchange rows and columns in a matrix. For instance, if A is the matrix

$$A = \begin{bmatrix} 4 & 5 & 6 \\ 3 & -1 & 0 \end{bmatrix},$$

then the transpose of A, denoted A^T is the matrix

$$A^T = \begin{bmatrix} 4 & 3 \\ 5 & -1 \\ 6 & 0 \end{bmatrix}.$$

Note that if A is an $m \times n$ matrix, then its transpose A^T is an $n \times m$ matrix. Also note that the transpose of the transpose is the original matrix, that is $(A^T)^T = A$. Also, note that transposition turns row vectors into column vectors, and vice versa.

Matrices in MATLAB. Last time we saw how easy it was to enter into MATLAB the coefficient matrix `A` and constant column vector `b` of a system of linear equations, and then compute the solution. Let's see what MATLAB does for the 3×2 system of equations above.

```
>> A = [5 2; 3 -1; 1 3]

A =
       5       2
       3      -1
       1       3

>> b = [12; 5; 5]

b =
      12
       5
       5

>> A\b

ans =
      2.0000
      1.0000

>>
```

Let's try MATLAB on this system

$$\begin{cases} 5x + 2y + z = 12 \\ 3x - y - z = 5 \end{cases}$$

```
>> A=[5 2 1;3 -1 -1]

A =
       5       2       1
       3      -1      -1
```

```
>> b=[12;5]

b =
    12
     5

>> A\b

ans =

    2.0000
    1.0000
         0
>>
```

Thus, $(x, y, z) = (2, 1, 0)$ is a solution to the system of equations. But it's not the only solution. Another is $(x, y, z) = (3, -7, 11)$, and there are infinitely more solutions to this indeterminate system.

On the other hand, the system

$$\begin{cases} 3x & + & 2y & = & 4 \\ 6x & + & 4y & = & 5 \end{cases}$$

is inconstant, that is, it has no solutions. MATLAB gives

```
>> A=[3 2;6 4]

A =
     3     2
     6     4

>> b=[4;5]

b =
     4
     5

>> A\b
Warning: Matrix is singular.

ans =
  -Inf
   Inf
>>
```

We'll see later how MATLAB can be used to help you find all the solutions to a system.

MATLAB also allows you to do create the identity matrices of various sizes, do addition and subtraction on matrices of the same shape, multiply a scalar and a matrix, take a transpose of a matrix, as well as many other operations, most of which we'll use throughout the course.

```
>> I = eye(3)

I =
     1     0     0
     0     1     0
     0     0     1

>> A=[4 5; 2 3; 0 1]

A =
     4     5
     2     3
     0     1

>> B=[1 1; 3 0; 4 4]

B =
     1     1
     3     0
     4     4

>> A+B

ans =
     5     6
     5     3
     4     5

>> 2*A-B

ans =
     7     9
     1     6
```

```
    -4    -2

>> B'

ans =
     1     3     4
     1     0     4

>>
```