Math 130 Linear Algebra

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Friday, 4 Sep 2009, third class

Due Today. Exercises from section 1.2: 1–2, 4–7 parts a–d each, 8, 9, T1, T5a.

No class Monday. Martin Luther King Jr. Day.

Read for Wednesday through section 1.4 on properties of matrix multiplication.

Due Wednesday. Exercises from section 1.3: 1–4, 7, 9, 11–12, 19–20, 33, T1, T4, ML1, ML2, ML5, ML6ac.

The ML exercises are Matlab exercises. Do them, but you don't have to make printed versions of them. Feel free to use Matlab to help you with your homework, even the parts of Matlab we haven't discussed in class.

Quiz Friday. Covering through section 1.3.

Due next Wednesday. Exercises from section 1.4: 11–13, 19, T.10, T.24, and T.30.

Last time. We introduced matrices and a few operations on them.

Today. Introduce vectors and their dot products, matrix multiplication, and briefly review summation notation.

Vectors. A *row vector* $[a_1 \quad a_2 \quad \ldots \quad a_n]$ is a $1 \times n$ matrix, that is a matrix with only one row. A *column vector* $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is a matrix with only one column. When the word "vector" is used, you can tell by the context whether we're talking about row or column vectors. Usually, we'll use column vectors. An n-vector is a vector with n entries, whether it's a row- or a column- vector. We usually identify vectors with n-tuples, $(a_1, a_2, \ldots, a_n)$. In our text and these notes, bold face lower case letters, like $\mathbf{u}$ and $\mathbf{v}$ are used. On the blackboard or paper, some people like to underline their vectors or put little arrows on top of them. I'll usually underline them.

The geometric interpretation of vectors. Since we can identify an n-vector with an n-tuple, and an n-tuple gives coordinates for a point in n-space, we can think of an n-vector as a point in n-space. For instance, the row-vector $[3 \quad 2]$ and the column-vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ can each be identified with the ordered pair $(3, 2)$ which describes a point in the (x, y)-plane which is 3 units to the right of the y-axis and 2 units above the x-axis.

But there's another interpretation that is more useful at times, and that is as an arrow between two points. In that interpretation, the vector $[3 \quad 2]$ describes any straight arrow from a point P in the plane to a point Q in the plane where the x-coordinate of Q is 3 more than the x-coordinate of P while the y-coordinate of P is 2 more than the y-coordinate of Q. For instance $[3 \quad 2]$ describes the arrow from $(0, 0)$ to $(3, 2)$, and it describes the arrow from $(1, 0)$ to $(4, 2)$, and infinitely many other parallel arrows.

We'll use both interpretations of vectors, either as points or as arrows, whichever interpretation is more useful to us at the time.

The dot product of two vectors. You can take the *dot product*, also called the *inner product*, of two vectors of the same length by multiplying their corresponding entries, then adding up all the products to get a scalar. For instance, the dot product of

$$\mathbf{u} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

is $\mathbf{u} \cdot \mathbf{v} = 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1 = 22$.

We'll interpret the geometric meaning of the dot product of two vectors in a later chapter. It will involve the length of the vectors and the angle between them. Right now, we can see that the dot product of a vector with itself, $\mathbf{u} \cdot \mathbf{u}$, is the square of the length of the vector.

When you're taking the dot product of two vectors, it doesn't matter whether they're row or column vectors. Sometimes we'll take the dot product of two row vectors, sometimes the dot product of two column vectors, and sometimes the dot product of a row vector and a column vector.

Example 3 illustrates a use of dot products.

Dot products can be described using summation notation. In general, the dot product of two vectors $\mathbf{v} = [v_1 \ v_2 \ \ldots \ v_n]$ and $\mathbf{w} = [w_1 \ w_2 \ \ldots \ w_n]$ is defined as

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

Rather than using the ellipsis, $\cdots$, which is meant to replace the missing terms, a summation notation is often used. In that notation the dot product is

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i.$$

You read this as "the sum as i goes from 1 through n of $v_i w_i$." It means that you form each of the n terms $v_i w_i$ for the n different values of i, then add those terms together. Thus, summation notation is a shorthand notation for adding sums of terms that have the same form.

Matrix multiplication. We've seen how to add and subtract matrices, and how to multiply them by scalars. Now we'll see how to multiply two matrices together.

You can multiply two matrices together that, in a way, generalizes dot products of vectors. What you do is treat each row of the first matrix as a row vector and treat each column of the second matrix as a column vector, and take all the dot products. The total number of dot products will be the number of rows of the first matrix times the number of columns of the second matrix. The results are all put together in one resulting matrix.

Take, for instance, the following two 3 by 3 matrices.

$$A = \begin{bmatrix} 4 & 5 & 6 \\ 3 & -1 & 0 \\ 2 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & 5 \\ -2 & -3 & 0 \end{bmatrix}$$

Think of A as being made of three row vectors and B as being made of three column vectors.

$$A = \left[\begin{array}{rrr} 4 & 5 & 6 \\ \hline 3 & -1 & 0 \\ \hline 2 & 0 & -2 \end{array}\right], \quad B = \left[\begin{array}{r|r|r} 2 & 1 & 1 \\ 0 & 4 & 5 \\ -2 & -3 & 0 \end{array}\right]$$

There are nine dot products to compute. For the first dot product, take the first row of A and the first column of B; you'll get $4 \cdot 2 + 5 \cdot 0 + 6 \cdot (-2) = -4$. That will give the upper left entry for the product matrix AB. The easiest way to see how it all works is to put the matrix A on the left the product matrix AB, and the matrix B above the product matrix AB as follows.

$$\begin{array}{cc} & \left[\begin{array}{r|r|r} 2 & 1 & 1 \\ 0 & 4 & 5 \\ -2 & -3 & 0 \end{array}\right] \\ \left[\begin{array}{rrr} 4 & 5 & 6 \\ \hline 3 & -1 & 0 \\ \hline 2 & 0 & -2 \end{array}\right] & \left[\begin{array}{r|r|r} -4 & 6 & 29 \\ \hline 6 & -1 & -2 \\ \hline 8 & 8 & 2 \end{array}\right] \end{array}$$

Each entry in the product matrix AB is found by taking the dot product of the row in A to its left with the column in B above it.

Products can be describes succinctly if we use summation notation. The entry c_{ik} in the i^{th} row and k^{th} column of the product matrix AB is

$$c_{ik} = \sum_j a_{ij} b_{jk}$$

(no kidding).

Example 11 gives a direct application of product matrices.

Example 10 shows that matrix multiplication is not always commutative. In that example $AB \neq BA$.

The size of matrices needed for multiplication. In order for there to be a product matrix AB, the length of a row in A has to be the same as the length of a column in B. If A is an m by n matrix, that means it has n entries in each row. So B will have to have n entries in each column, in other words, B has to have n rows. Then B is an n by k matrix (where k is the number of columns that B has, whatever that is). Thus, you can multiply an m by n matrix with an n by k matrix. The result will be an m by k matrix.

Systems of linear equations are linear matrix equations. We'll have a lot of uses for matrix multiplication as the course progresses. Here's a way we can use it right now.

Take, for example, the system of equations

$$\begin{aligned} 5x + 2y &= 12 \\ 3x - y &= 5 \\ x + 3y &= 5 \end{aligned}$$

Let A be the coefficient matrix for this system, so that

$$A = \begin{bmatrix} 5 & 2 \\ 3 & -1 \\ 1 & 3 \end{bmatrix},$$

and let $\mathbf{b}$ be the constant matrix (a column vector) for this system, so that

$$\mathbf{b} = \begin{bmatrix} 12 \\ 5 \\ 5 \end{bmatrix}.$$

Finally, let $\mathbf{x}$ be the variable matrix for this system, that is, a matrix (another column vector) with the variables as its entries, so that

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then the original system of equations is described by the matrix multiplication $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 5 & 2 \\ 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \\ 5 \end{bmatrix}$$

In general, each system of linear equations corresponds to a single matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where A is the matrix of coefficients in the system of equations, $\mathbf{x}$ is a vector of the variables in the equations, and $\mathbf{b}$ is a vector of the constants in the equations. This interpretation allows us to interpret something rather complicated, namely a whole system of equations, as a single equation.

Matrix products in Matlab. If `A` and `B` are two matrices of the right size, that is, `A` has the same number of columns that `B` has rows, then the expression `A*B` gives their product. You can compute powers of square matrices as well. If `A` is a square matrix, then `A^3` computes the same thing as `A*A*A`. See section 12.3 of our textbook for more details.

If `x` and `y` are two vectors of the same length, then `dot(x,y)` computes their dot product. If they're both column vectors, then that gives the same thing as `x'*y`.