Math 130 Linear Algebra

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Read for Monday section 1.6 which goes into the solutions of linear systems of equations in a little more detail than we already have.

Due Monday. Exercises from section 1.4: 11–13, 19, T.10, T.24, and T.30.

Due next Friday. Exercises from section 1.5: 1–2, 5–6, 15–17.

Last time. We looked at more properties of the matrix operations, powers of matrices, symmetric and skew symmetric matrices.

Quiz Today. Covering through section 1.3.

Today. Finish discussion started on Wednesday. If we have time, we'll begin a discussion on the important concept of linear transformations, also called matrix transformations, in section 1.5.

Matrix transformations. One of the most important uses of matrices is for describing linear transformations of the plane $\mathbf{R}^2$, of space $\mathbf{R}^3$, and of higher dimensional spaces $\mathbf{R}^n$. Before getting to the general definition of linear transformations, we'll look at a few in the plane and space.

Geometric vectors. We'll represent an n-vector as a straight arrow in $\mathbf{R}^n$. Today, we'll always take one end of the vector, the tail, to be the origin $(0, 0, \ldots, 0)$. Then, the head of the vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the point $(x_1, x_2, \ldots, x_n)$ in $\mathbf{R}^n$. For example, we'll draw the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as an arrow that starts at the origin $(0, 0)$ in the plane $\mathbf{R}^2$ and ends at the point $(3, 2)$. This way we identify a column vector with an ordered pair, so we're treating vectors as points.

Square 2×2 matrices describe linear transformations of the plane. For example, consider the 2×2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now, take a generic point (x, y) in the plane, but treat it as the vector $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$. We can form the matrix product $A\mathbf{u}$, since the number of rows of A equals the number of columns of $\mathbf{u}$, to get another vector $\mathbf{v} = A\mathbf{u}$.

$$\mathbf{v} = A\mathbf{u} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

Thus, the matrix A transforms the point (x, y) to the point $f(x, y) = (x, -y)$. You'll recognize this right away as a reflection across the x-axis.

Every 2×2 matrix describes some kind of geometric transformation of the plane. But since the origin $(0, 0)$ is always sent to itself, not every geometric transformation can be described by a matrix in this way. Only the ones that we'll call linear transformations can be described by a matrix. There are a few others we'll look at.

Rotations. The matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

determines the transformation that sends the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to the vector $\begin{bmatrix} -y \\ x \end{bmatrix}$. In particular, the two "basis" vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are sent to the vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, respectively. Note that the first column of the 2×2 matrix says where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ goes while the second column says where $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. You'll recognize this transformation as a rotation around the origin by 90°. (Our convention is to always take counterclockwise rotations to be by a positive number of degrees, but clockwise ones by a negative number of degrees.)

Rotations by other angles θ can be described with the help of trig functions. The matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

describes a rotation of the plane by an angle of θ.

For example, the matrix that describes a rotation of the plane around the origin of 10° counterclockwise is

$$\begin{bmatrix} \cos 10° & -\sin 10° \\ \sin 10° & \cos 10° \end{bmatrix} = \begin{bmatrix} 0.9848 & -0.1736 \\ 0.1736 & 0.9848 \end{bmatrix}$$

since $\sin 10° = 0.1736$ and $\cos 10° = 0.9848$.

All rotations preserve distance. That means that the distance between any two vectors $\mathbf{u}$ and $\mathbf{v}$ is the same as the distance between their images $A\mathbf{u}$ and $A\mathbf{v}$. Such transformations are called *rigid transformations* or *isometries.*

Reflections. We've already seen that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ describes a reflection across the x-axis. Likewise, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ describes a reflection across the y-axis. There's a 2×2 matrix for reflection across any line through the origin. What matrix describes a reflection across the line $y = x$?

Note that reflections, like rotations, are transformations that preserve distance, that is, they're isometries, too.

Contractions and expansions. Not all linear transformations preserve distance. For instance, contractions and expansions don't. The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ sends a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to the vector $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$. Thus, every point is sent twice as far away from the origin. That's an expansion by a factor of 2. Every scalar matrix where the scalar is greater than 1 describes an expansion.

When the scalar is between 0 and 1, then the matrix describes a contraction. For instance $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ moves points toward the origin half as far away as where they started.

The particular scalar matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ sends a point to the other side of the origin, but the same distance away from the origin. That's the same as a 180° rotation. These are sometimes called *half turns* or *point inversions.*

Other scalar matrices with negative scalars describe transformations that can be thought of as compositions of point inversions and either expansions or contractions.

Sometimes the term "dilatation" is used for any of these transformations determined by scalar matrices.

Other transformations. Not every linear transformation of the plane belongs to one of the classes described above. For example, the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ describes a "shear transformation" that fixes the

x-axis, moves points in the upper half-plane to the right, but moves points in the lower half-plane to the left.

Another interesting transformation is described by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$ Which sends the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to the vector $\begin{bmatrix} 2x \\ y/2 \end{bmatrix}$. The plane is transformed by stretching horizontally by a factor of 2 at the same time as it's squeezed vertically.

These last two examples are plane transformations that preserve areas of figures, but don't preserve distance.

If you randomly choose a two by two matrix, it probably describes a linear transformation that doesn't preserve distance and doesn't preserve area.

Transformation of $\mathbf{R}^3$. A 3×3 matrix describes a transformation of space. There are many kinds of such transformations, some isometries, some not. Some isometries are (1) reflections across planes that pass through the origin and (2) rotations around lines that pass through the origin, but there are others. Some transformations that aren't isometries include dilations, contractions, shears, but there are many others.

Linear transformations between spaces. An m by n rectangular matrix describes a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$. We'll use the notation $\mathbf{R}^n \to \mathbf{R}^m$ to indicate that $\mathbf{R}^n$ is the *domain* of the transformation and $\mathbf{R}^m$ is the *range* or *codomain* of the transformation.

Here, for instance, is a 2×3 matrix that describes a linear transformation $\mathbf{R}^3 \to \mathbf{R}^2$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. It sends the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in the domain $\mathbf{R}^3$ to the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in the codomain $\mathbf{R}^2$. This particular transformation is a "projection" from 3-space onto the xy-plane that forgets the z-coordinate.

A linear transformation $\mathbf{R}^n \to \mathbf{R}$ is described by a row matrix. For example, the matrix $\begin{bmatrix} 1 & 3 & -2 & -4 \end{bmatrix}$ describes the linear transformation $\mathbf{R}^4 \to \mathbf{R}$ which sends the point (w, x, y, z) to $w + 3x - 2y - 4z$.

Later in the course we'll define a linear transformation $L : \mathbf{R}^n \to \mathbf{R}^m$ to be a function that preserves addition of vectors

$$L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$$

and multiplication by scalars

$$L(c\mathbf{u}) = cL(\mathbf{u}).$$

We'll see then that every linear transformation $L : \mathbf{R}^n \to \mathbf{R}^m$ can be represented by a unique $m \times n$ matrix A, that is, $L(\mathbf{u}) = A\mathbf{u}$ for each vector $\mathbf{u} \in \mathbf{R}^n$.