Math 130 Linear Algebra

Prof. D. Joyce, Clark University

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Read for Friday. Section 1.7 on inverse matrices.

Due Friday. Exercises from section 1.5: 1–2, 5–6, 15–17.

Due Monday. Exercises from section 1.6: 1–8, 13–14, 20, 28, T8.

Last time. Linear transformations, also called matrix transformations.

Today. Solutions of linear systems of equations. We discussed this before on the first day, and you're already familiar with it from high school courses, so we won't spend too much time on it. Know what "echelon form" means, what "reduced echelon form" means, and how to solve systems of linear equations. We'll look at an example of a system where the solution is not unique so you can see how you can describe the set of its solutions.

We may also look at the preview of graph theory on page 48.

Elementary row operations, and row echelon and reduced row echelon form for a matrix. When you want to solve a system of linear equations $A\mathbf{x} = \mathbf{b}$, form the augmented matrix by appending the column $\mathbf{b}$ to the right of the coefficient matrix A. Then you can solve the system of equations by operating on the rows of the augmented matrix rather than on the actual equations in the system. The three *row operations* are those operations on a matrix that don't change the solution set of the corresponding system of linear equations. The simplest ones are called elementary row operations, and the elementary ones are of three types.

1. Exchange two rows.
2. Multiply or divide a row by a nonzero constant.
3. Add or subtract a multiple of one row from another.

You can use these three operations to convert the augmented matrix to a particularly simple form that allows you to read off the solutions. The idea is to eliminate the variables from the equations so that each variable only occurs once. If there's a unique solution, that works perfectly. If the system is indeterminate and has infinitely many solutions, it doesn't work perfectly, but we'll see how that works.

You can systematize this elimination process to form an algorithm. Work from the leftmost column right one column at a time to simplify the augmented matrix (and, therefore, the system it represents). The column you're working on is called the *pivot column*. Look down the column for the first nonzero entry that has all 0s to its left. Call that entry the *pivot element*, or more simply the *pivot*. There are three things to do corresponding to the three elementary row operations.

1. Exchange the pivot row with the highest row that has 0s in the pivot column and all columns to the left. (Do nothing if there is no such row.)
2. Divide the pivot row by the pivot so that the value of the pivot becomes 1.

3. Subtract multiples of the pivot row from the other rows to clear out the pivot column (except the pivot itself).

When you're all done with this algorithm, the matrix will be in something called reduced row echelon form. The word echelon comes from a particular stairstep formation of troops. For us, a matrix is in *reduced row echelon form* if

1. the rows of all zeros (if any) appear at the bottom of the matrix
2. the first nonzero entry of a nonzero row is 1
3. that leading 1 appears appears to the right of leading 1s in higher rows
4. all the other entries in a column that has a leading 1 are 0

If the first three conditions hold, the matrix is said to be in *row echelon form.*

The reduction algorithm to convert an augmented matrix to reduced row echelon form goes by the name Gauss-Jordan reduction. The partial algorithm that stops with a row echelon form goes by the name Gaussian elimination. As we've seen, this reduction was known to the ancient Chinese.

Two martrices are *row equivalent* if their corresponding systems of linear equations have the same solutions. It's easy to show that means you can perform a sequence of elementary row operations on one to eventually get the other.

Homogeneous systems. A system of linear equations is called *homogeneous* if all the constant terms are 0, that is, it is of the form $A\mathbf{x} = \mathbf{0}$. Every homogeneous system has at least one solution, namely the trivial solution where all the variables have the value 0. The question we have for homogeneous systems is whether they have any nontrivial solutions.

Row reduction in MATLAB**.** Section 12.4 of our text shows how you can perform the elementary row operations in MATLAB, either by explicitly manipulating the rows of a matrix or by using the routine `reduce`.

If you're only interested in the resulting reduced row eschelon form of a matrix, then you can use the command `rref`. For example, here it's used for the system of linear equations

$$\begin{cases} 2v + w - 3x + 4y + 5z = 7 \\ v - 2w + 2x + 3y = 4 \\ 4v + 5x + 2y + z = -4 \end{cases}$$

```
>> A=[2 1 -3 4 5 7;1 -2 2 3 0 4;4 0 5 2 1 -4]

A =
     2     1    -3     4     5     7
     1    -2     2     3     0     4
     4     0     5     2     1    -4

>> rref(A)

ans =
    1     0     0     1.5366     1.3171     1.8049
    0     1     0    -1.5610    -0.1951    -3.3415
    0     0     1    -0.8293    -0.8537    -2.2439
```

From that reduced row eschelon form, we can determine the general solution. For that y and z may be freely chosen, and

$$\begin{aligned} v &= -1.5366y - 1.3171z + 1.8049 \\ w &= 1.5610y + 0.1951z - 3.3415 \\ x &= 0.8293y + 0.8537z - 2.2439 \end{aligned}$$