Math 130 Linear Algebra

Prof. D. Joyce, Clark University

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Due today. Exercises from section 1.6: 1–8, 13–14, 20, 28, T8.

Due Friday. Exercises from section 1.7: 1–4, 9, 11, 12, 17a, 22, T4, T5, ML1–ML4.

Last time. The concept, the definition, and theorems about inverse matrices. A method for finding the inverse of a matrix.

Next time. Start chapter 3 on determinants.

Applications. Which sections should we look at from Chapter 2?

First test. Check your schedule. We'll select a date for the first test next meeting.

Today. Proof that the method for finding inverses we saw last time works.

A proof that the method works. The question now is: why does this method works? In other words what is a proof of it? Kolman and Hill give one explanation. Here's another.

The conversion of $[A|I]$ to reduced echelon form is performed with a series of elementary row operations. These elementary row operations come in three types, and each type can be thought of as multiplying the matrix by a certain square matrix, called an elementary matrix, on the left. For instance, one operation is to exchange rows r and s of the matrix. That operation is performed by multiplying the matrix on the left by a square matrix E, the elementary matrix, which looks just like the identity matrix I except rows r and s of I are exchanges. For instance, the matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

exchanges the second and third rows of a $3 \times n$ matrix. So EA is the same as A except its second and third rows are exchanged.

Likewise, there is a square matrix to multiply the rth row by a nonzero constant c. That's just the identity matrix where c replaces the 1 in the rth row. For instance, the matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

doubles the third row of a matrix. So EA is the same as A except the third row of EA is twice the third row of A.

Finally, there is a square matrix to add a multiple of one row to another. For example, to subtract twice the first row from the second, just multiply on the left by this matrix:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Before going on, notice that each of these three kinds of elementary row operations is invertible, so the elementary matrices associated to each are also invertible.

Now, suppose the sequence of elementary row operations performed on the matrix $[A|I]$ to put it into reduced echelon form is described by the elementary matrices $E_1, E_2, \ldots, E_s$. After the first elementary operation, the matrix $[A|I]$ is converted to $[E_1A|E_1I]$. After the second, it becomes $[E_2E_1A|E_2E_1I]$. And after the last it looks like $[E_s \cdots E_2E_1A|E_s \cdots E_2E_1I]$. Let's denote the product of the E_is as E, that is, $E = E_s \cdots E_2E_1$. Then we have $[EA|E]$ as the reduced echelon form.

Now, we have two cases to consider. One is where the first half, EA, of the $n \times 2n$ matrix that's in reduced echelon form has a 1 in each row. The other is where some row has no one in it but is all 0s. If any row is all 0s, then the last row will be all 0s.

In the first case where there's a 1 in each row of the square matrix EA, because it's in reduced echelon form, therefore it has to be the identity. That says $EA = I$. Since E is the product $E_s \cdots E_2E_1$ of invertible elementary matrices, therefore E is also invertible. Multiply the equation $EA = I$ by E^{-1} on the left to get $A = E^{-1}$, and therefore $A^{-1} = E$. It is in this first case that A is invertible and you can find the inverse.

In the second case, the last row of EA is all 0s. We need to show that A cannot have an inverse in this case. We'll suppose A does have an inverse and derive a contradiction. Now, the matrix EA has all 0s in its last row, so if you multiply it on the right by any matrix, then the product will also have all 0s in its last row. But A is invertible, so multiply EA on the right by $A^{-1}E^{-1}$. The result is I, and I doesn't have 0s in its last row, a contradiction. Therefore, A doesn't have an inverse in this second case.

That finishes the proof that this method will either construct an inverse matrix or show that the inverse doesn't exist.