Math 130 Linear Algebra

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Due today. Exercises from section 3.1. Do parts a and b only of all of these exercises: 1–6, 15, 19, and 20.

Due Friday. Exercises from section 3.1: T1, T6, and T12.

First test. Next Monday 5 Oct, to cover through section 3.1.

Later assignment. Exercises from section 3.2: 1, 3ab, 4ab, 10ab, 11ab, 20, 23.

Last time. Some properties of determinants including

- A matrix and its transpose have the same determinant, that is, $|A| = |A^T|$.
- If two rows of a matrix are interchanged, then the determinant is negated. Likewise, if two columns of a matrix are interchanges, then the determinant is also negated.
- If two rows of a matrix are equal, then the determinant of that matrix is 0. Likewise, if two columns in a matrix are the same, then its determinant is 0.
- If a matrix has all 0s in a row, then its determinant is 0. Likewise, if a matrix has a 0 column, then its determinant is 0.
- If you multiply every element in a row of a matrix by the same number, then the determinant is multiplied by that number. That is, if every entry in one row of the matrix A is multiplies by c to get the matrix B, then $|B| = c|A|$. Likewise, if you multiply a column by a constant, then the determinant is multiplied by a constant.
- If one row of a matrix is a multiple of another, then the determinant is 0. Likewise for columns.
- Multilinearity of the determinant.
- Adding a multiple of one row to another doesn't change the determinant.

Today. We'll look at Determinants of diagonal and triangular matrices, determinants of products, and determinants of inverses. Then we'll see an efficient way to compute determinants. Finally, we'll start looking at another way to define determinants called *cofactor expansion.* We won't spend very much time on it because we've already got a good way to find determinants. We'll see soon how to use this cofactor expansion to find the inverse of a matrix, and another way to solve a system of equations, called *Cramer's rule.*

Diagonal and triangular matrices. It's very easy to compute the determinants of diagonal and triangular matrices. That's because every one of the $n!$ terms is 0 except the term which is the product of the diagonal entries. The determinant of a diagonal or triangular matrix is just the product of its diagonal entries.

One very important special case of this is that the determinant of an identity matrix is 1, that is, $|I| = 1$.

An efficient method for computing determinants. Except for small n, you don't want to use the definition of determinant to compute a determinant. There's just too much work. You have to compute $n!$ terms, each term being the product of n entries. That's $n!\,(n-1)$ multiplications to compute. There are better ways.

The three elementary row operations each do something to the determinant. Exchanging rows negates the determinant. Multiplying a row by a constant multiplies the determinant by that constant. Adding a multiple of one row to another doesn't change the determinant at all.

Using these row operations, you can fairly quickly convert a matrix to triangular form. The determinant of the resulting matrix is just the product of the elements down the diagonal.

Multiplying matrices. The determinant of the product of two matrices is the product of the determinants of the matrices. $|AB| = |A|\,|B|$.

We'll skip the proof of this statement. It can be shown from the multilinearity property stated above, but the proof is long.

Determinants of inverses, and nonsingularity. It follows immediately (from the multiplicative property just stated) that if a matrix is invertible, then the determinant of its inverse is the reciprocal of its determinant

$$|A^{-1}| = |A|^{-1}$$

since $|A|\,|A^{-1}| = |AA^{-1}| = |I| = 1$. Therefore, a matrix is invertible if and only if its determinant is not 0.

An example for cofactor expansion. Probably the best way to understand cofactor expansion is to use it in an example. In this example, we'll do cofactor expansion across the first row of a matrix, but you can do it for any row or column. Let's start with a typical 3×3 matrix that we want to invert.

$$A = \begin{bmatrix} 5 & 2 & -1 \\ 4 & -2 & 3 \\ 6 & 1 & 4 \end{bmatrix}$$

For each of the three entries across the first row, we'll cross out the row and column of that entry to get a 2×2 matrix. The determinant of that 2×2 matrix is called a *minor*. Then we'll multiply each entry times its minor and add or subtract the resulting product to get the determinant. That gives $|A| =$

$$5 \cdot \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 3 \\ 6 & 4 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 4 & -2 \\ 6 & 1 \end{vmatrix}$$

Thus, we've reduced the problem of finding one 3×3 determinant to finding three 2×2 determinants. The rest of the computation goes like this:

$$\begin{aligned} |A| &= 5 \cdot (-11) - 2 \cdot (-2) + (-1) \cdot 16 \\ &= -55 + 4 - 16 = -67 \end{aligned}$$

Now, notice how we alternately added and subtracted the three terms to get the product. When you expand across the first row, the signs you use are $+-+$. But if you expand across the second row, you'll use the signs $-+-$. The sign you use for the ij^{th} entry is $(-1)^{i+j}$. This corresponds to this checkerboard pattern for the entries of the matrix

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

which starts with $+$ in the upper left corner.

The ij^{th} *cofactor*, denoted A_{ij}, of a matrix $A = [a_{ij}]$ is the product of this sign and the ij^{th} minor.

In order to prove that this cofactor expansion works, you'd probably use induction on the size n of the matrix, although you could expand it completely out and show that the resulting expression with $n!$ terms is the same expression we used to define determinants in terms of permutations.

Cofactor expansion isn't a practical method for computing determinants. But it is useful to prove things about matrices.

The adjoint matrix and inverses. The adjoint matrix of a square matrix A, denoted $\text{adj}\,(A)$, is the transpose of the matrix of cofactors. That is, the ij^{th} entry of $\text{adj}\,(A)$ is the cofactor A_{ji}.

The adjoint matrix, the original matrix, and the determinant are related in a special way, namely, the product of the adjoint matrix and the original matrix, in either order, is a diagonal matrix with $|A|$ down the diagonal. That is,

$$A\,(\text{adj}\,(A)) = (\text{adj}\,(A))\,A = |A|\,I.$$

This gives us a way to find inverses. If you divide both sides by the determinant, $|A|$, you get

$$A\left(\frac{1}{|A|}\text{adj}\,(A)\right) = \left(\frac{1}{|A|}\text{adj}\,(A)\right)A = I,$$

which shows that $\dfrac{1}{|A|}$ adj is the inverse of A.

Again, this is not a practical method for finding inverses, but it is helpful in other ways.

Cramer's rule. This is a method based on cofactors to find the solution to a system of n equations in n unknowns when there is exactly one solution. The solution is has the determinant in the denominator, and the only time the determinant is not zero is when there's a unique solution.

Here's an example to show how to apply Cramer's rule. Let's suppose we have the following system of three equations in three unknowns.

$$\begin{array}{rcrcrcr} x & + & y & + & 3z & = & 6 \\ 2x & + & 3y & - & 4z & = & -2 \\ 3x & - & 2y & + & 5z & = & 7 \end{array}$$

First, compute the determinant Δ of the 3×3 coefficient matrix.

$$\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & 5 \end{vmatrix} = -54$$

Next, replace the first column by the constant vector, and compute that determinant.

$$\Delta_x = \begin{vmatrix} 6 & 1 & 3 \\ -2 & 3 & -4 \\ 7 & -2 & 5 \end{vmatrix} = -27$$

Then in the unique solution, $x = \Delta_x/\Delta = \frac{1}{2}$. Next, replace the second column by the constant vector, and compute that determinant.

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & -2 & -4 \\ 3 & 7 & 5 \end{vmatrix} = -54$$

So $y = \Delta_y/\Delta = 1$. Likewise, replace the third column by the constant vector.

$$\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 3 & -2 \\ 3 & -2 & 7 \end{vmatrix} = -81$$

which gives $z = \frac{3}{2}$. Thus, the unique solution is $(x, y, z) = (\frac{1}{2}, 1, \frac{3}{2})$.