Math 130 Linear Algebra

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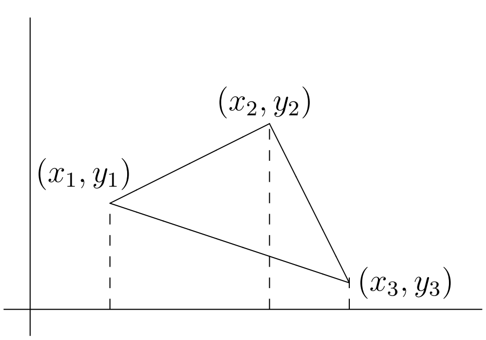
Due today. Exercises from section 4.1: 3, 5, 9ab, 13, 19, 21ab, 22ab.

Due Friday. Exercises from section 4.1: 23, 24, 27, 28, T5, T7.

Last time. Vectors in the plane $\mathbf{R}^2$; length $\|\mathbf{v}\|$ of a vector; geometric interpretations of addition, negation, subtraction, and inner products of vectors; properties of dot products; and orthogonal vectors.

Today. Areas of triangles and parallelograms, unit vectors, vectors in dimension n, and coordinates for physical 3-space.

Areas of triangles and parallelograms in terms of determinants. As described in the text, you can determine the area of a triangle in the plane with vertices at $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3(x_3, y_3)$.

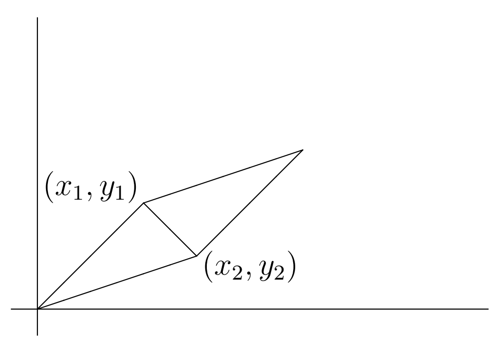


If the vertices are at the position shown, where they go around the triangle in a clockwise direction, you can determine the area by adding the areas of two trapezoids with tops P_1P_2 and P_2P_3, then subtracting the area of the trapezoid with top P_3P_1. A little algebra shows that the resulting area is half the absolute value of the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Note that if you exchange the names of two of the vertices, then the value of the determinant is negated. Right now, we're not interested in the sign of the determinant, but that sign becomes very important because it can be used to tell whether the orientation is clockwise or counterclockwise. When the sign is included with the area, then this is called the *signed area* of the triangle.

Likewise, the area of a parallelogram where any three of its vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is the absolute value of that determinant.

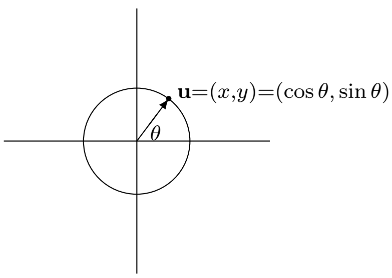


Note that if the vertex (x_3, y_3) is placed at the origin $(0, 0)$, then the determinant simplifies to

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

That gives us a geometric interpretation of the 2×2 determinant as the signed area a certain parallelogram with two sides being the vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ which are the columns of the matrix.

Unit vectors. A *unit vector* is a vector whose length is 1. If a unit vector $\mathbf{u}$ is placed in standard position with its tail at the origin, then it's head will land on the unit circle $x^2+y^2=1$. Every point on the unit circle (x,y) is of the form $(\cos\theta, \sin\theta)$ where θ is the angle measured from the positive x-axis in the counterclockwise direction.



Thus, every unit vector in the plane is of the form

$$\mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

We can interpret unit vectors as being directions, and we can use them in place of angles since they carry the same information as an angle.

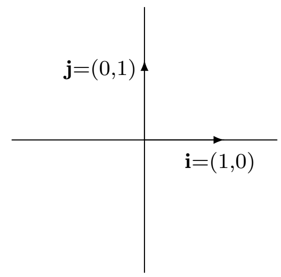
When we move on to three dimensions, we'll still use unit vectors and they will still signify directions, but they carry more information than just one angle. After all, if you want to name a point on a sphere, you need to give two angles, longitude and latitude.

Now that we have unit vectors, we can treat every vector $\mathbf{v}$ as a length and a direction. The length of $\mathbf{v}$ is $\|\mathbf{v}\|$, of course. And its direction is the unit vector $\mathbf{u}$ in the same direction which can be found by

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

The vector $\mathbf{v}$ can be reconstituted from its length and direction by multiplying $\mathbf{v} = \|\mathbf{v}\|\,\mathbf{u}$.

Standard unit vectors i and j for the plane $\mathbf{R}^2$. Our coordinatized plane $\mathbf{R}^2$ has two standard directions, the x-direction and the y-direction, and we can encode them as unit vectors. We'll denote the unit vector in the x-direction as $\mathbf{i}$ so that $\mathbf{i} = (1,0)$, and the unit vector in the y-direction, as $\mathbf{j}$ so that $\mathbf{j} = (0,1)$.



Every vector $\mathbf{v} = (x,y)$ can be uniquely written as a linear combination of these two standard unit vectors.

$$\mathbf{v} = (x,y) = x\mathbf{i} + y\mathbf{i}$$

Vectors in dimension n and n-space $\mathbf{R}^n$. So far, we've studied vectors in $\mathbf{R}^2$, primarily because we can draw them easily. But everything we've said about dimension 2 also holds in an arbitrary dimension n.

The vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ can be interpreted as an arrow in n-space $\mathbf{R}^n$ with a certain length and a certain direction. As in the case when $n=2$, it can be interpreted as lots of different arrows with that length and direction. When it's put in "standard position," the head of the arrow is at the point $(v_1, v_2, \ldots, v_n)$ and the tail of the arrow is at the origin $(0,0,\ldots,0)$. Using this standard position, we can identify a vector with a point $(v_1, v_2, \ldots, v_n)$.

Vector operations. The primary two vector operations are vector addition and scalar multiplication. Vector addition, $+ : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, is an operation that takes two vectors $\mathbf{v}$ and $\mathbf{w}$ in n-space and produces another vector $\mathbf{v}+\mathbf{w}$ in n-space. Scalar multiplication, $\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, takes a scalar c, that is, a real number, and a vector $\mathbf{w}$ in n-space and produces another vector $c\mathbf{w}$ in n-space. These two operations enjoy the same properties for n-space as they do for 2-space.

Later on in the course, we'll define an abstract vector space as a set equipped with two operations, called vector addition and scalar multiplication, that satisfy these same properties.

Other operations can be defined from these two, namely, negation $-\mathbf{v} = (-1)\mathbf{v}$, and subtraction $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-1)\mathbf{w}$, and they also have the usual properties.

Coordinates for physical space. Frequently, people interpret physical space as $\mathbf{R}^3$. Of course, we know now that the geometry of physical space is not the same as $\mathbf{R}^3$, but close enough for many practical purposes. In order to place a coordinate system on physical space, several choices are required. Different choices lead to different coordinate systems.

(a). Choose a location in physical space to call the origin, $(0, 0, 0)$.

(b). Choose a line through the origin to be the x-axis.

(c). Choose a point on the x-axis to be $(1, 0, 0)$. This choice determines the scale of the coordinate system. The distance between $(0, 0, 0)$ and $(1, 0, 0)$ will be the unit distance.

(d). Choose a line perpendicular to the x-axis to be the y-axis. There are infinitely many to choose from, but they all lie in a plane perpendicular to the x-axis passing through the origin.

(e). There are two points on the y-axis at unit distance from the origin. Choose one of them to be the point $(0, 1, 0)$.

(f). There is one line perpendicular to both the x-axis and the y-axis. Call it the z-axis.

(g). There are two points on this z-axis at unit distance from the origin. Choose one of them to be the point $(0, 0, 1)$. Depending on which one you choose, the resulting coordinate system is called a right-handed coordinate system or a left-handed coordinate system for physical space. (See the text for a diagram.)