Math 130 Linear Algebra

Prof. D. Joyce, Clark University

Monday, 26 Oct 2009

Return quiz.

Due today. Exercises from section 4.3: 1, 4–7, 21, 22.

Due Wednesday. Exercises from section 4.3: 25, 26, 27, 29, T9, T10, T11.

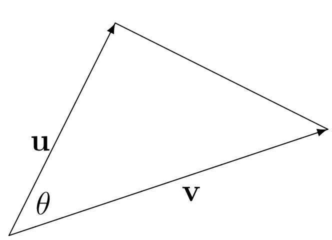
Due Friday. Exercises from section section 5.1: 1ab, 2ab, 9-12, T2, T3, T4, T5, (T7), ML1–2, ML5–6.

Last time. Cross Products in $\mathbf{R}^3$: their definition, properties, and length; and triple scalar products.

For next time. Read section 5.2 on lines and planes.

Today. The cross product and triangles, parallelograms, and parallelepipeds.

Area of a triangle in $\mathbf{R}^3$. Consider a triangle in 3-space where two of the sides are $\mathbf{u}$ and $\mathbf{v}$.

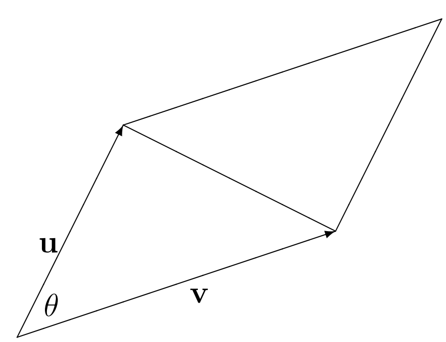


Taking $\mathbf{u}$ to be the base of the triangle, then the height of the triangle is $\|\mathbf{v}\| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Therefore, the area of this triangle is

$$\text{Area} = \tfrac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = \tfrac{1}{2}\|\mathbf{u}\times\mathbf{v}\|.$$

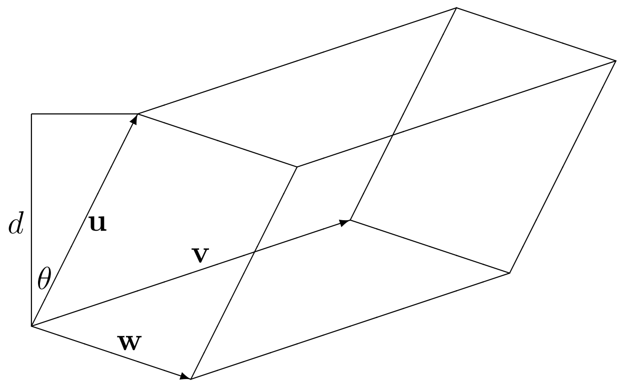
(In general, the area of a any triangle is half the product of two adjacent sides and the sine of the angle between them.)

Area of a parallelogram in $\mathbf{R}^3$. Now consider a parallelogram in 3-space where two of the sides are $\mathbf{u}$ and $\mathbf{v}$.



Of course, if the triangle is doubled to a parallelogram, then the area of the parallelogram is $\|\mathbf{u}\times\mathbf{v}\|$.

Volume of a parallelepiped in $\mathbf{R}^3$. Next consider a parallelepiped whose edges are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.



Take the parallelogram with sides **v** and **w** as the base of this solid. This base has area $\|\mathbf{v}\times\mathbf{w}\|$. The cross product vector $\mathbf{v}\times\mathbf{w}$ is a vector orthogonal to the plane of **v** and **w**. It's not shown in the diagram, but the line labelled d is in the same direction. The height of this solid is the distance d from this face to the one parallel to it. Since we have a right triangle with angle θ, adjacent side d, and hypotenuse $\|\mathbf{u}\|$, therefore

$$d = \|\mathbf{u}\|\,|\cos\theta|,$$

where, this time, θ is the angle between the vectors **u** and $\mathbf{v}\times\mathbf{w}$. Therefore the volume of the parallelepiped is

$$\text{Volume} = \|\mathbf{v}\times\mathbf{w}\|\,\|\mathbf{u}\|\,|\cos\theta| = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$$

which is the absolute value of the triple product

$$[\mathbf{u},\mathbf{v},\mathbf{w}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Cross products and triple scalar products in Matlab. The function `cross` will compute cross products, and the matrix building ability along with computing determinants will compute triple scalar products.

```
>> u = [1 3 5]
u =
     1     3     5

>> v = [0 2 -1]
v =
     0     2    -1

>> cross(u,v)
ans =
   -13     1     2

>> norm(cross(u,v))
ans =
   13.1909

>> w = [2 0 3]
w =
     2     0     3

>> t = [u;v;w]
t =
     1     3     5
     0     2    -1
     2     0     3

>> det(t)
ans =
   -20
```

Thus, the cross product $\mathbf{u}\times\mathbf{v}$ is $(-13, 1, 2)$, and since the length of that cross product is 13.1909, that's the area of a parallelgram with sides **u** and **v**.

The triple product $[\mathbf{u},\mathbf{v},\mathbf{w}]$ is -20 so the volume of the parallelopiped with edges **u**, **v**, and **w** is 20.