Math 130 Linear Algebra

Prof. D. Joyce, Clark University

Friday, 30 Oct 2009

Due today. Exercises from section section 5.1: 1ab, 2ab, 9-12, T2, T3, T4, T5, (T7), ML1–2, ML5–6.

Due Monday. Exercises from section 5.2: 5ab, 6ab, 10ab, 13, T3.

Due Wednesday. Exercises from section 6.1: 11, 15, 20, T3.

Last time. Lines and planes in 3-space.

For next time. Read section 6.2 on subspaces.

Today. Finish discussion from last time.

Begin chapter 6 on abstract vector spaces. The first section 6.1 defines what is meant by an abstract real vector space with a precise definition and several examples.

The concept of abstract vector space. By now, we know real n-space, $\mathbf{R}^n$, pretty well. We abstract from it to make a general definition of vector space that includes $\mathbf{R}^n$ and some other things as well. Real n-space has a lot of structure, some of which we'll require a vector space to have, but some we won't require. We won't require, for instance, that a vector space have a coordinate system. We will require that a vector space have an operation of vector addition and an operation of scalar multiplication. But we won't require that a vector space have a distance function that assigns a distance $\|\mathbf{v}\|$ to a vector. Also, we won't require that a vector space have a dot product operation or a cross product operation.

Thus, for our vector spaces we abstract two of the operations, vector addition and scalar multiplication, from $\mathbf{R}^n$, but we ignore the rest of the structure that $\mathbf{R}^n$ has.

The precise definition. A *vector space* is defined to be a set V, whose elements we will call *vectors*, equipped with two operations, the first called *vector addition*, which takes two vectors $\mathbf{v}$ and $\mathbf{w}$ and yields another vector, usually denoted $\mathbf{v}+\mathbf{w}$, and the second called *scalar multiplication*, which takes a real number c and a vector $\mathbf{v}$ and returns another vector, usually denoted $c\mathbf{v}$, such that the following properties (called axioms) all hold

a. Vector addition is commutative: $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$ for all vectors $\mathbf{v}$ and $\mathbf{w}$;

b. Vector addition is associative: $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$;

c. There is a vector, denoted $\mathbf{0}$ and called the *zero vector*, such that $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$ for each vector $\mathbf{v}$;

d. For each vector $\mathbf{v}$, there is another vector, denoted $-\mathbf{v}$ and called the *negation* of $\mathbf{v}$, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$;

e. Scalar multiplication distributes (on the left) over vector addition: $c(\mathbf{v}+\mathbf{w})=c\mathbf{v}+c\mathbf{w}$ for each real number c and all vectors $\mathbf{v}$ and $\mathbf{w}$;

f. Scalar multiplication distributes (on the right) over real addition: $(c+d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$ for all real numbers c and d and each vector $\mathbf{v}$;

g. Multiplication and scalar multiplication associate: $c(d\mathbf{v}) = (cd)\mathbf{v}$ for for all real numbers c and d and each vector $\mathbf{v}$;

h. 1 acts as the identity for scalar multiplication: $1\mathbf{v} = \mathbf{v}$ for each vector $\mathbf{v}$.

That's a long definition, but it has to be long if we want an abstract vector space to have all the properties that $\mathbf{R}^n$ has, at least with respect to vector addition and scalar multiplication.

Lots of other properties follow from these axioms, such as $0\mathbf{v} = \mathbf{0}$, and we'll discuss those properties next time.

Examples. Of course n-space, $\mathbf{R}^n$, is a vector space. But what other vector spaces are there? Quite a few. Most will be finite dimensional like $\mathbf{R}^n$ is, but some will be infinite dimensional. (We haven't yet defined dimension, but we will, and we'll define it in such a way that n-space has dimension n.) Also, most of the other examples we'll look at have some other structure besides the structure for being vector spaces. That other structure isn't needed for the examples to be vector spaces.

Polynomials. Fix a positive integer n. Let P_n be the set of all polynomials of degree n or less with real coefficients in the variable t. For instance, $5t^4+3t^2-7$ is a polynomial of degree 4, so it's an element of P_4, and it's an element of all P_n for $n \geq 4$, too. This set P_n has an operation of addition because we can add two polynomials to get another, and we can multiply any polynomial by a real constant. So P_n has the two operations required to be a vector space. Furthermore, every one of the required properties hold for these two operations (since they're just addition and multiplication), so P_n is a vector space.

Note that P_n doesn't have a coordinate system (but it isn't hard to give it one), and P_n doesn't have a distance function or a dot product operation, but it's still a vector space.

When we finally define the dimension of P_n, we'll see its dimension is $n+1$. (Why $n+1$? Why isn't its dimension n?)

What happens when there's no limit on the degree of the polynomial? Let P be the set of all polynomials with real coefficients in the variable t (but no limitations on the degree of t). This P is also a vector space, but it's infinite dimensional. Note that P has more structure than it needs to be a vector space; you can also multiply two polynomials to get another polynomial.

Functions. We can generalize the last example P to all functions. Let F be the set of all real-valued functions with domain all of $\mathbf{R}$. Then, since we can add two functions, and we can multiply a function by a real constant, and all the properties hold, this F is another vector space.

There are many modifications you can make to get related vector spaces. For instance, you can consider functions defined on a particular interval $[a,b]$. Or you could consider only continuous functions. Or you could consider only differentiable functions. Each one of these variants is a vector space.

Matrices. Fix m and n and consider the set M_{mn} of all $m \times n$ matrices. Since we can add two matrices, and we can multiply any matrix by a real number, we've got the two operations needed for a vector space, and they satisfy all the axioms, so M_{mn} is a vector space. As you can guess, it will turn out that the dimension of the vector space M_{mn} is mn.

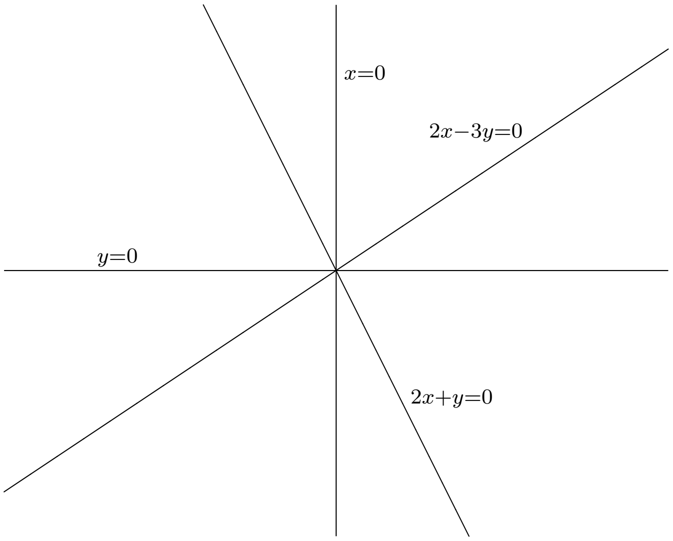
Note that when $m = n$, there's another operation, matrix multiplication, but that isn't needed for M_{nn} to be a vector space.

Vector spaces over fields other than the real numbers. The definition given above for vector spaces is actually a definition for *real vectors spaces*, that is, vector spaces over $\mathbf{R}$. That's because all the scalars are to be real numbers. But there are other possibilities besides real numbers for scalars. If you allow the scalars to be complex numbers, then the definition becomes vectors spaces over **C**, the field of complex numbers. We'll have use for them in chapter 8.

Our text talks quite a bit about bit vectors. They're vectors whose components are each either 0 or 1 with the unusual addition that $1 + 1 = 0$. These vectors are in vector spaces over the field of 2 elements, sometimes denoted $GF(2)$. (There are fields $GF(p)$ for each prime number p.)

You can have vector spaces over any field. All that's needed to have a field is a set equipped with operations of addition, subtraction, multiplication, and division with the usual properties. Thus, **R**, **C**, and $GF(2)$ are fields. Another example of a field is the field **Q** of rational numbers.

Subspaces. We will be particularly interested in subspaces of vector spaces. For instance, the set of vectors of the form $(3t, 2t)$ is a subspace of $\mathbf{R}^2$.



It's the line $2x - 3y = 0$ through the origin. It's a vector space because the sum of any two vectors in it is another vector in it, and a scalar multiple of any vector in it is another vector in it. The diagram shows four 1-dimensional subspaces of the plane. There's the line $2x - 3y = 0$, the line $2x + y = 0$, the x-axis $y = 0$, and the y-axis $x = 0$.

There's much more on subspaces to come later.