Math 130 Linear Algebra

Prof. D. Joyce, Clark University

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Due today. Exercises from section 6.1: 11, 15, 20, T3.

Due Friday. Exercises from section 6.2: 1-3, 5, 6, 9, 14, 17-19.

Due Monday. Exercises from section 6.2: 23–25, 27, T4, T10, ML3, ML5.

Second test. Select date.

Last time. Properties of vector spaces, subspaces.

Today. Subspaces of 3-space, solution spaces, spans. (We might begin talking about isomorphism of vector spaces, too. See the notes for next time.)

Coming up. The concept of span is one of two concepts we'll need for the definition of the dimension of a vector space. The other concept is that of *linear independence*, which comes next in section 64. We'll use the two concepts together for a general definition of *basis* of a vector space. (It will be a subset of vectors that span the whole vector space, but they're independent.) Then the *dimension* of the vector space will be defined to be the number of elements in a basis. (Just before we make that definition, we'll have to prove that every basis of the vector space has the same number of elements.)

Subspaces of 3-space $\mathbf{R}^3$. You can probably guess what all the subspaces of 3-space are. And you'd be right. Let's describe what they are, but not prove anything right now. First, there's the trivial subspace $\mathbf{0}$ that consists of the origin alone. Next, each line through the origin is a subspace. Next each plane through the origin is a subspace, Finally, the entire 3-space is a subspace of itself.

When we look at subspaces of $\mathbf{R}^n$, we'll find subspaces of of dimension 0, 1, and so forth up through n. There will be just one subspace of dimension 0, namely the trivial subspace $\mathbf{0}$, and there will be just one subspace of dimension n, namely all of $\mathbf{R}^n$, but there will be infinitely many subspaces for each intermediate dimension.

Solution spaces of homogeneous systems of linear equations. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of equations where $\mathbf{x}$ is a vector of n unknowns. Let V be the set of solutions of this system, so that $\mathbf{x} \in V$ if and only if $A\mathbf{x} = \mathbf{0}$.

Now, let's show that V is a subspace of $\mathbf{R}^n$. (1) $\mathbf{0} \in V$ since $A\mathbf{0} = \mathbf{0}$. (2) V is closed under vector addition, since if $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$, then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}$. (3) V is closed under scalar multiplication since if $A\mathbf{x} = \mathbf{0}$, then $A(c\mathbf{x}) = cA\mathbf{x} = \mathbf{0}$.

In fact, this solution space is one of subspaces of $\mathbf{R}^n$ mentioned before. One of the questions we'll have about the solution space is how big is it? That is, what is its dimension? That will turn out to be the number of degrees of freedom in specifying a solution of the system $A\mathbf{x} = \mathbf{0}$.

The solution space of a homogeneous system $A\mathbf{x} = \mathbf{0}$ is also called the *null space* of the matrix A.

The span of a set of vectors. Since vector spaces are closed under linear combinations, we should have a name for the set of all linear combinations of a given set of vectors, and that will be their *span*.

Definition 1. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space V. The *span* of S, written span S or span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, is the set of all linear combinations of vectors in S. That is, span S consists of all vectors of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_k.$$

The proof of the following theorem is left for you to prove in an exercise for this section.

Theorem 2. The span of a set S is a subspace of V.

You can also describe span S as the smallest subspace of V that contains all of S. That's because it *is* a vector space that contains all of S, but it only has linear combinations of vectors in S, so every vector in it has to be in every vector space that contains all of S.

The linear combination problem in Matlab**.** Consider the question whether a particular vector $\mathbf{v}$ is a linear combination of given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. This is the same question as: Is $\mathbf{v}$ in the span of the given vectors?

This question can be solved in Matlab. After all, you're just looking to solve the vector equation

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_k$$

for the unknowns $c_1, c_2, \ldots, c_n$, and that's just a system of linear equations. See section 12.7 of the text for more details.

Here we'll determine if the vector $\mathbf{v} = (1, 4, 2, 6)$ is a linear combination of the vectors $\mathbf{x} = (3, -1, 1, 0)$, $\mathbf{y} = (1, 1, 1, 1)$, and $\mathbf{z} = (1, 2, -1, 3)$. Note that the system will have 4 equations (one for each coordinate) in three unknowns (being c_1, c_2, and c_3), so we don't expect it to have a solution. Treat all the vectors as column vectors, place the vectors as columns in an augmented matrix, and row reduce it using the function `rref`. (In fact, I'll enter them as rows, then transpose.)

```
>> aug = [3 -1 1 0;1 1 1 1;1 2 -1 3;1 4 2 6]'

aug =
     3     1     1     1
    -1     1     2     4
     1     1    -1     2
     0     1     3     6

>> rref(aug)

ans =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

Thus the four equations are inconsistent since the last equation says $0 = 1$. Thus, $\mathbf{v}$ is not a linear combination of the others.