Math 130 Linear Algebra

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Friday, 6 Nov 2009

Due today. Exercises from section 6.2: 1-3, 5, 6, 9, 14, 17-19.

Due Monday. Exercises from section 6.2: 23–25, 27, T4, T10, ML3, ML5.

Due Wednesday. Exercises from section 6.3: 1, 2, 6–10, ML1, ML2.

Second test. Next Friday. Covers 3.2, 4.1–4.3, 5.1–5.2, and 6.1–6.2.

Last time. Subspaces of 3-space, solution spaces, spans.

Today. Isomorphism, duality; concepts of linear independence and basis. We'll continue the study of bases next time.

Isomorphism of vector spaces and dual spaces. These are two abstract concepts that we'll briefly look at.

Isomorphisms. Recall how we usually treat an n-vector as a column vector, but sometimes as a row vector. We'll see how both are correct, but we'll also find a way of distinguishing them. For now, let's fix vectors in $\mathbf{R}^n$ as column vectors.

Let's use the notation $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m)$ for the set of linear transformations $\mathbf{R}^n \to \mathbf{R}^m$. We've seen how to identity a linear transformation T with a particular matrix A_T in $M_{m\times n}$. Thus, $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m)$ is "isomorphic" to $M_{m\times n}$. Intuitively speaking, two things are isomorphic are the same thing but the names of the elements are different. We'll make this concept of isomorphism more precise as we go on.

Both $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m)$ and $M_{m\times n}$ are vector spaces over $\mathbf{R}$. Furthermore, the addition of two linear transformations corresponds to the addition of matrices, that is, if A_1 is the matrix that represents the linear transformation T_1, and A_2 the matrix for T_2, then the matrix for $T_1 + T_2$ is $A_1 + A_2$. Also, multiplication by scalars corresponds: if A_T represents T, and c is a scalar, then the matrix cA_T represents the linear transformation cT. Thus, the isomorpism $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m) \cong M_{m\times n}$ also respects the vector space structure on these two vector spaces. We say $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m)$ and $M_{m\times n}$ are isomorphic as vector spaces.

Here's a more precise, but more abstract, definition definition of the isomorphism $V \cong W$ of two vector spaces: there are two linear transformations $T_1 : V \to W$ and $T_2 : W \to V$ whose compositions are identity transformations, that is, $T_1 \circ T_2$ is the identity transformation on W while $T_2 \circ T_1$ is the identity transformation on V.

The dual of a vector space. If V is any vector space over $\mathbf{R}$, we'll define the space *dual* to V as $\mathrm{Hom}(V, \mathbf{R})$, and denote it V^*.

Let's look at the space $(\mathbf{R}^n)^*$ dual to $\mathbf{R}^n$. It is just $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R})$ which is isomorphic to $M_{1\times n}$. Thus, an element of $(\mathbf{R}^n)^*$ is a linear transformation $\mathbf{R}^n \to \mathbf{R}$, but we can identify it with a row vector of length n. We've taken the vectors in $\mathbf{R}^n$ to be column vectors, and then the vectors in the dual space $(\mathbf{R}^n)^*$ are row vectors.

We can treat transposition of a column matrix as a way of associating each element (column vector) of $\mathbf{R}^n$ with the corresponding row vector in the dual space $(\mathbf{R}^n)^*$.

We won't go into it now, but we could take transposition, $M_{m\times n} \to M_{n\times m}$, which associates each $m \times n$ matrix to a corresponding $n \times m$ matrix as giving an isomorphism $\mathrm{Hom}(R^n, R^m) \to \mathrm{Hom}(R_m, R_n)$, that is, the vector space of linear transformations $\mathbf{R}^n \to \mathbf{R}^m$ is dual to the vector space of lienar transformations $\mathbf{R}^m \to \mathbf{R}^n$.

Spanning sets of vector spaces. Recall that last time we defined the span of a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V to be the set of all linear combinations

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_k$$

of vectors in S. Then you showed span S was a subspace of V.

When span S is all of V, we also say that S *spans* V. We are particularly interested in finding spanning sets of V.

How do you know if S spans V? Just see if every vector $\mathbf{v}$ is some linear combination of the vectors in S.

Linear independence. The question of spanning a vector space asks if you have enough vectors in a set S to get all other vectors in a space as a linear combination of the vectors in S. The question of independence asks if you have too many, that is, can you do without some of them because they're redundant.

Definition 1. A set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$$

of vectors in a vector space V is said to be *linearly dependent* if there are scalars $c_1, c_2, \ldots, c_k$ not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

You can read this as saying that at least one of the vectors is a linear combination of the rest, for if $c_i \neq 0$, then $\mathbf{v}_i$ is a linear combination of the rest.

If the vectors aren't linearly dependent, then we say they're *linearly independent*. In other words, no vector in S is a linear combination of the others.

A logically equivalent statement is that S is linearly independent if the only way a linear combination of vectors in S can equal 0,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

is when each of the scalars $c_1, c_2, \ldots, c_k$ are all 0. In other words, $\mathbf{0}$ is not a nontrivial linear combination of the vectors in S.

How do you know whether the vectors in S are linearly dependent or independent? Just solve the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ for $c_1, c_2, \ldots, c_k$. This single vector equation is a system of homogeneous linear equations. If you only get the trivial solution, then the vectors in S are linearly independent. If you get any other solution, then they're dependent.

For $\mathbf{R}^n$, the standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are linearly independent. You can see that each of them, $\mathbf{e}_i$, is the only one of them with a nonzero i^{th} coordinate, therefore it is not a linear combination of the rest. So they're all independent.

In general, two vectors $\mathbf{v}$ and $\mathbf{w}$ are linearly independent if and only if each is not a multiple of the other. Geometrically that means they do not lie on the same line through the origin $\mathbf{0}$.

Testing for linear independence using MATLAB**.** In order to tell if vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are independent, check to see if the homogeneous system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ has any nontrivial solutions. We can do that in MATLAB with the `rref` function. See section 12.7 for more details.

For example, let's see see if the vectors $(1, 3, 5, 7)$, $(2, 0, 1, 3)$, $(-1, 2, -1, 0)$, and $(4, 3, 1, -5)$ are independent. Place them in four columns of a coefficient matrix, and row reduce the matrix.

```
>>A=[1 3 5 7;2 0 1 3; -1 2 -1 0;-5 17 9 15]'

A =
```

```
     1     2    -1    -5
     3     0     2    17
     5     1    -1     9
     7     3     0    15

>> rref(A)

ans =

     1     0     0     3
     0     1     0    -2
     0     0     1     4
     0     0     0     0
```

There are nontrivial solutions. The unknown c_4 can be chosen freely, and the general solution is $(c_1, c_2, c_3, c_4) = (-3c_4, 2c_4, -4c_4, c_4)$. Thus they are not independent. Taking $c_4 = -1$, we can write $\mathbf{0}$ as a nontrivial comination of the four vectors by

$$3\mathbf{v}_1 - 2\mathbf{v}_2 + 4\mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0}.$$

Basis of a vector space. Now, we'll combine the two concepts of span and linear independence together to give a definition of basis.

Definition 2. A subset S of a vector space V is a *basis* of V if each vector in V may be uniquely represented as a linear combination of vectors from S.

Theorem 3. A subset S of a vector space V is a basis if and only if (1) S spans V, and (2) S is linearly independent.

Proof. Part I. Suppose S is a basis by the definition. Then every vector is a linear combination, so S spans V. Also, the vector $\mathbf{0}$ is uniquely a linear combination of elements of S, so S is linearly independent.

Part II. Suppose that S spans V and it's linearly independent. Since it spans V, every vector can be represented as some linear combination of elements of S. We have yet to show there's only one such linear combination. Suppose that a vector $\mathbf{v}$ can be represented in two ways:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k.$$

Then

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots + (c_k - d_k)\mathbf{v}_k.$$

By linear independence, each $c_i = d_i$. Hence the two representations were the same. Q.E.D.

For an example of a basis, the standard basis $\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \mathbf{e}_n = (0, 0, \ldots, 1)$ is a basis of the vector space $\mathbf{R}^n$. It is not the only basis of $\mathbf{R}^n$, however. If n vectors are chosen at random from $\mathbf{R}^n$, there is a very good chance they will both span $\mathbf{R}^n$ and be independent, and so they'll be a basis, too.

Theorem 4. Every finite spanning subset S of a vector space V has a subset T which is a basis of V.

Proof. Let T be one of the largest subsets of S that is linearly independent. First, we'll show that any element $s \in S$ is a linear combination of vectors in T. Either is an element of T, in which case it's a linear combination, or it is not an element of T, which means that $T \cup \{s\}$ is a larger subset of S, therefore not linearly independent, therefore s is a linear combination from T. Thus, every element in S is a linear combination from T. But S spans V, therefore T spans V. Since T is linearly independent, therefore T is a basis of V. Q.E.D.

This proof was not constructive, but you can find a construction based on it. Go through the elements of S one at a time. Each time you find a vector that is linearly dependent on the preceding ones, throw it out, but keep all the rest.