Math 130 Linear Algebra Quiz, Sep 2009

Scale. 9–10 A, 7–8 B, 5–6 C.

Problem 1. [5] Display a system of 3 linear equations in 3 unknowns that has no solutions. (You don't have to try to solve the system; just write the three equations.)

Some thought has to go into constructing such a system. If you write down a random linear system of 3 equations in 3 unknowns, it will, with very high probablility, have a unique solution.

There are trivial examples of a system with no solutions, for instance, $x = 0$, $x = 1$, and $x = 2$. Although only x appears explicitly, it can be interpreted as an equation in three unknowns where the coefficients of the other two unknowns are 0. You can also find more interesting examples.

One way is to make the third equation contradict the first two. Take any system of 2 equations in 3 unknowns, then add a third inconsistent equation by adding the first two equations together (or any linear combination of the first two, for that matter) and then changing the constant. For example,

$$\begin{array}{rcrcrcr} 2x & + & 3y & + & 5z & = & 10 \\ 5x & - & 3y & + & 3z & = & 1 \end{array}$$

is a system of 2 equations in 3 unknowns. So, here's a system of 3 equations in 3 unknowns with no solutions:

$$\begin{array}{rcrcrcr} 2x & + & 3y & + & 5z & = & 10 \\ 5x & - & 3y & + & 3z & = & 1 \\ 7x & & & + & 8z & = & 12 \end{array}$$

Problem 2. [5] A square matrix A is said to be *symmetric* if it equals its own transpose, that is, $A^T = A$. In other words, for all indices i and j, $a_{ij} = a_{ji}$. For example,

$$\begin{bmatrix} 3 & 0 & 5 \\ 0 & 2 & 7 \\ 5 & 7 & -2 \end{bmatrix}$$

is a 3×3 symmetric matrix. Give an example where the product of symmetric matrices does not commute. That is, display two symmetric matrices A and B, and compute both products, AB and BA, which should not be equal. (Hint: 2×2 matrices are large enough.)

It turns out that almost any two randomly chosen symmetric matrices don't commute. Although it wasn't asked for, let's figure out when they do commute in the 2×2 case.

Let A be an arbitrary 2 by 2 symmetric matrix. It is of the form $A = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$. And let B be another of them. $B = \begin{bmatrix} s & t \\ t & u \end{bmatrix}$. First, compute AB, then compute BA.

$$AB = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} s & t \\ t & u \end{bmatrix} = \begin{bmatrix} xs + yt & xt + yu \\ ys + zt & yt + zu \end{bmatrix}$$

$$BA = \begin{bmatrix} s & t \\ t & u \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} sx + ty & sy + tz \\ tx + uy & ty + uz \end{bmatrix}$$

Note that the entries on the diagonals agree for the two product matrices. Also, the upper right entry of each equals the lower left entry of the other. That means, for AB to equal BA, it is necessary and sufficient that a single equation be satisfied, namely,

$$xt + yu = ys + zt.$$

When you choose random values for x, y, z, s, t, and u, it is not very likely that this equation turns out true. (One case where it is true is when $x = z$ and $x = u$, that is, when the upper left entries equal the lower right entries.)