Curves and paths
Math 131 Multivariate Calculus

D Joyce, Spring 2014

Fermat and Descartes developed coordinate geometry in the first half of the 1600s. They gave coordinates (x, y) for points in the plane and made the correspondence of a curve in the plane with an equation in x and y. That made the connection between geometry and algebra.

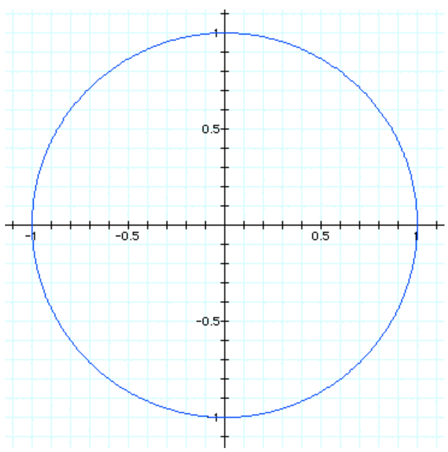

Figure 1: The unit circle

For example, the unit circle, that is the circle of radius 1 centered at the origin, has the equation $x^2 + y^2 = 1$.

When we move up to three dimensions, you can't describe a curve by an equation in x, y, and z. Instead, an equation in three variables describes a surface in $\mathbf{R}^3$. Some other way is needed to describe a curve in $\mathbf{R}^3$, and we can do that by parameterizing the curve. Such a parameterization is useful for curves in $\mathbf{R}^2$ as well.

Paths. A function $\mathbf{x} : \mathbf{R} \to \mathbf{R}^m$ is called a *path*. It can be thought of as the path of a point $\mathbf{x}$ moving in $\mathbf{R}^m$. A path parameterizes a curve.

We'll distinguish between a path and the curve it travels. A path is given by a function $\mathbf{x} : \mathbf{R} \to \mathbf{R}^m$, and we'll usually take the variable to be t suggesting time. The curve is the image of this path, that is, a subset of $\mathbf{R}^m$.

Examples 1 (Unit circle)**.** For instance, $\mathbf{v} : \mathbf{R} \to \mathbf{R}^2$

$$\mathbf{x}(t) = (\cos t, \sin t)$$

describes the path of a point that moves in the plane $\mathbf{R}^2$ as t changes. The curve it traverses is the circle $x^2 + y^2 = 1$. A different function $\mathbf{x} : \mathbf{R} \to \mathbf{R}^2$ where

$$\mathbf{x}(t) = (\cos 2t, \sin 2t)$$

also describes a path of a point that moves around the same circle, but the point is moving twice as fast. The curve is the same for both functions.

Paths are also called *parameterizations* of curves, where the parameter is the independent variable, t.

Velocity, acceleration, and speed. You've probably already studied parametric curves, that is, paths of points, before in calculus of one variable. They're appropriate to study there since there is only one independent variable, t. When you studied them you may have also looked at the velocity, acceleration, and speed of the moving point.

The velocity of a moving point is just the derivative of its position, but when there's more than one dimension, velocity isn't just positive or negative, but has a direction associated to it. For the circle example where the position $\mathbf{x}$ at time t is given by $\mathbf{x}(t) = (\cos t, \sin t)$, the velocity is

$$\mathbf{x}'(t) = (-\sin t, \cos t)$$

where each coordinate is the derivative of the corresponding coordinate of $\mathbf{x}(t)$. The velocity is a vector with both a direction and a magnitude, also called its norm or length. That magnitude is called *speed*. Thus, the speed in this example is

$$\|\mathbf{x}'(t)\| = \|(-\sin t, \cos t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1,$$

so the point is moving with constant speed. Although its speed is constant, its velocity is changing since the direction is changing.

The acceleration of a moving point is the derivative of its velocity, that is, the second derivative of its position. For the circle example, that's

$$\mathbf{x}''(t) = (-\cos t, -\sin t).$$

What's special about this particular example that acceleration is the negation of position. That means that this point travelling at a constant speed around the circle is always accelerating towards the center of the circle.

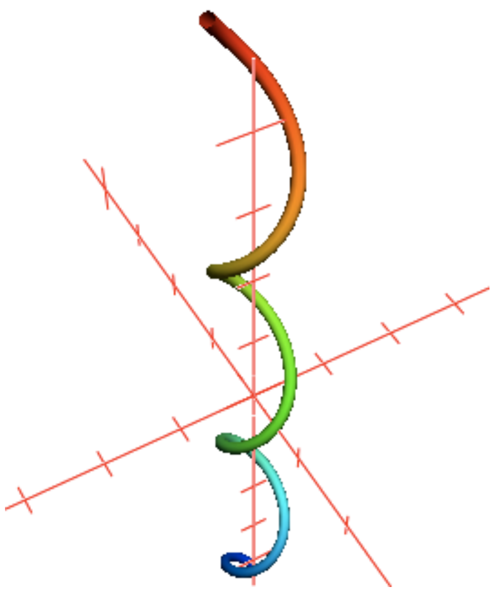

Figure 2: Helix

Points can move in higher dimensional space as well.

Example 2 (The helix)**.** Consider the helix in 3-space in figure 2. The term helix refers to spiral in 3-space. The vector valued function $\mathbf{x} : \mathbf{R} \to \mathbf{R}^3$ given by

$$\mathbf{x}(t) = (\cos t, \sin t, t)$$

describes a point travelling along a helix. The function $\mathbf{x}$ describes a parameterization of the helix with the parameter t.

Whereas $\mathbf{x}(t)$ describes the position of a point at time t, $\mathbf{x}'(t)$ describes its velocity, $\|\mathbf{x}(t)\|$ describes its speed, and $\mathbf{x}''(t)$ describes its acceleration. In this example these have the following values.

$$\begin{aligned} \mathbf{x}'(t) &= (-\sin t, \cos t, 1) \\ \|\mathbf{x}(t)\| &= \sqrt{\sin^2 t + \cos^2 + 1^2} = \sqrt{2} \\ \mathbf{x}''(t) &= (-\cos t, -\sin t, 0 \end{aligned}$$

Curves in dimensions greater than two are usually described by parameterizations like this, whereas curves in dimension two can either be given either by a parameterization or by a single equation in x and y.

Math 131 Home Page at
`http://math.clarku.edu/~djoyce/ma131/`