# Proof of Green's theorem
Math 131 Multivariate Calculus

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**Summary** of the discussion so far.

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

Green's theorem can be interpreted as a planer case of Stokes' theorem

$$\oint_{\partial D} \mathbf{F} \cdot ds = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA.$$

In words, that says the integral of the vector field $\mathbf{F}$ around the boundary $\partial D$ equals the integral of the curl of $\mathbf{F}$ over the region $D$. In the next chapter we'll study Stokes' theorem in 3-space.

Green's theorem implies the divergence theorem in the plane.

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\,ds = \iint_D \nabla \cdot \mathbf{F}\,dA.$$

It says that the integral around the boundary $\partial D$ of the the normal component of the vector field $\mathbf{F}$ equals the double integral over the region $D$ of the divergence of $\mathbf{F}$.

**Proof of Green's theorem.** We'll show why Green's theorem is true for elementary regions $D$. These regions can be patched together to give more general regions.

First, suppose that $D$ is a region of "type 1," that is, it can be described by inequalities $a \le x \le b$ and $\gamma(x) \le y \le \delta(x)$ where $\gamma$ and $\delta$ are $C^1$ functions. First, we'll show that

$$\iint_D -\frac{\partial M}{\partial y}\,dA = \oint_{\partial D} M(x,y)\,dx$$

We can directly integrate the left integral as a double integral

$$\begin{aligned}
&\iint_D -\frac{\partial M}{\partial y}\,dA \\
&= \int_a^b \int_{\gamma(x)}^{\delta(x)} -\frac{\partial M}{\partial y}\,dy\,dx \\
&= \int_a^b -M(x,y)\Big|_{y=\gamma(x)}^{\delta(x)}\,dx \\
&= \int_a^b \big( M(x,\gamma(x)) - M(x,\delta(x)) \big)\,dx \\
&= \int_a^b M(t,\gamma(t))\,dt - \int_a^b (t,\delta(t))\,dt
\end{aligned}$$

We're almost done. The first integral

$$\int_a^b M(t,\gamma(t))\,dt$$

is the path integral along the curve $y = \gamma(x)$ from left to right, that is, it is $\displaystyle\int_\gamma M(x,y)\,dx$. Likewise, the second integral $\displaystyle\int_a^b (t,\delta(t))\,dt$ is the parameterization along the curve $y = \delta(x)$ from left to right, but that portion of the boundary $\partial D$ should go from right to left, and the minus sign reverses the orientation. The two vertical sides $x = a$ and $x = b$ of $D$ form the other two parts of $\partial D$. Since $\dfrac{dx}{dt}$ is 0 on those vertical paths, therefore

$$\int M(x,y)\,dx = \int M(x,y)\frac{dx}{dt}\,dt = \int 0\,dt = 0$$

along them. Therefore,

$$\int_a^b M(t,\gamma(t))\,dt - \int_a^b (t,\delta(t))\,dt = \oint_{\partial D} M(x,y)\,dx$$

Likewise, if $D$ is a region of "type 2," that is, one bounded between horizontal lines, then

$$\iint_D \frac{\partial N}{\partial x}\,dA = \oint_{\partial D} N(x,y)\,dy$$

where the minus sign is dropped because the symmetry exchanging $y$ for $x$ reverses orientation.

Adding these two equations gives Green's theorem for $D$

$$\iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \oint_{\partial D} M(x,y)\,dx + N(x,y)\,dy$$

That takes care of the case when the region is of both type 1 and type 2. But regions that can be decomposed into a finite number of these can be patched together to take care of the general case.

Q.E.D.

There are other proofs that are more inclusive to show that some regions that are a union of an infinite number of these regions also satisfy Green's theorem.

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