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Higher-order partial derivatives
Math 131 Multivariate Calculus

D Joyce, Spring 2014

Higher-order derivatives. Let's start with a function $f : \mathbf{R}^2 \to \mathbf{R}$ and only consider its second-order partial derivatives. Take, for example,

$$f(x, y) = (x + y)e^y.$$

We can easily compute its two first-order partial derivatives.

$$f_x = \frac{\partial f}{\partial x} = e^y$$

$$f_y = \frac{\partial f}{\partial y} = e^y + (x + y)e^y = (1 + x + y)e^y$$

Each of these two functions, in turn, has two partial derivatives. Their partials with respect to x are

$$f_{xx} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = 0$$

$$f_{yx} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = e^y$$

Notice how the two different notations for partials indicate the order of taking derivatives differently. The partials with respect to y are

$$f_{xy} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = e^y$$

$$f_{yy} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = e^y + (1+x+y)e^y = (2+x+y)e^y$$

Thus, there are four second-order partial derivatives of this function f. Notice that in this example the two mixed partial derivatives are equal, that is, $f_{yx} = f_{xy}$. These two mixed partials are usually equal. We won't prove the following theorem, but we'll frequently use it.

Theorem 1. When the first and second partials of a function of two variables x and y are all continuous functions, the order that the partials are taken doesn't matter, that is, $f_{yx} = f_{xy}$, or in partial notation

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}.$$

Functions that have continuous first and second partial derivatives are called $\mathcal{C}^2$ functions, or functions of class $\mathcal{C}^2$.

The concept of second-order derivatives generalizes as you would expect it to for third- and higher-order derivatives, and not just to functions $\mathbf{R}^2 \to \mathbf{R}$ but to scalar-valued functions $\mathbf{R}^n \to \mathbf{R}$ in general. The text has several examples, and you'll be working out more in the exercises.

Preview of an application of second-order partial derivatives. You'll recall from calculus of one variable that second derivatives were used in something called the second-derivative test for extreme values. You used it on critical points, that is, values of x where $f'(x)$ was 0. It said that if the second derivative was positive at a critical point, then f had a minimum there, but if the second derivative was negative, then a maximum.

Things are more complicated in higher dimensions since there are more directions to consider. All the second order partial derivatives come into play. A particular matrix called the *Hessian* is filled with them, and it's used to determine the character of critical points.

Minimal surfaces, another application A surface with a given boundary is called a *minimal surface* if among all surfaces with that boundary it has the least area. Of course, a circle bounds its interior, but that's not interesting since it's planar. We're interested in the non planar ones. The first two were the catenoid and helicoid.

An interesting minimal surface is Costa's, illustrated in figure 1. It was constructed in 1982 by Celso José da Costa. Its boundary consists of three

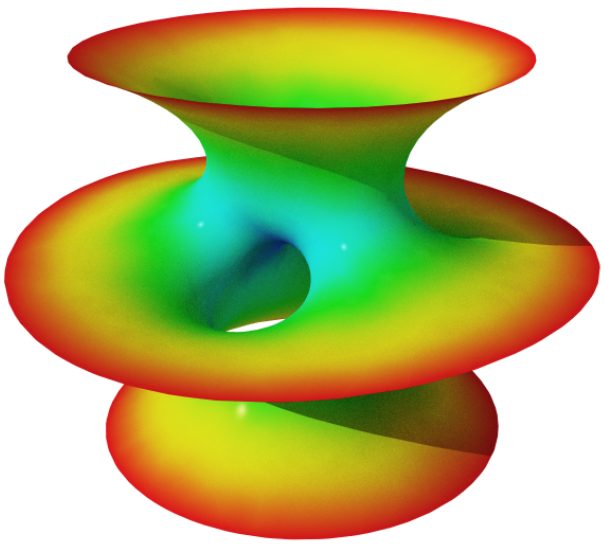

Figure 1: Costa's minimal surface

circular components. See the wiki article Costa's minimal surface for details about it.

You can make your own minimal surfaces by dipping bent clothes hangers in a soapy solution, but don't let there be any bubbles. A spherical bubble isn't a minimal surface.

It turns out that if a minimal surface is the graph $z = f(x, y)$ of a function two variables, then

$$(1 + z_y)^2 z_{xx} = (1 + z_x^2) z_{yy}$$

and conversely, functions whose derivatives satisfy that partial differential equation have graphs that are minimal surfaces. One such graph is *Scherk's surface.*

Example 2 (Scherk's surface)**.** Heinrich Scherk constructed some embedded minimal surfaces in 1834. This surface has the equation

$$e^z \cos y = \cos x.$$

The part of the surface above the square $[-1.5, 1.5] \times [-1.5, 1.5]$ is illustrated in figure 2. As an exercise, verify that $(1 + z_y)^2 z_{xx} = (1 + z_x^2) z_{yy}$ holds for this surface.

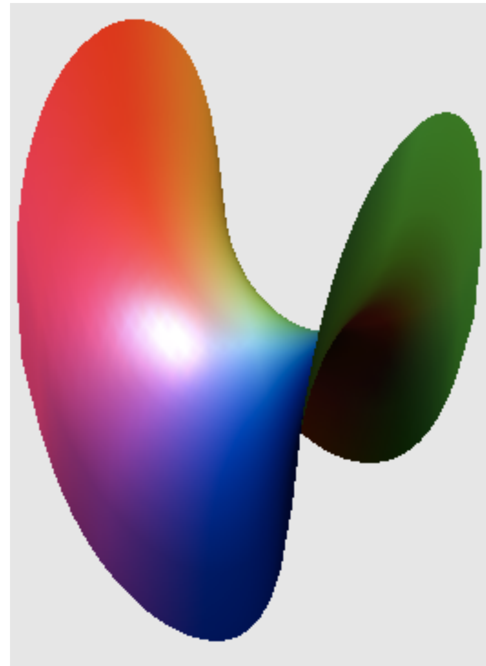

Figure 2: Scherk's surface

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