Things you need to know about linear algebra
Math 131 Multivariate Calculus
D Joyce, Spring 2014

The relation between linear algebra and multivariate calculus. We'll spend the first few meetings reviewing linear algebra. We'll look at just about everything in chapter 1.

Linear algebra is the study of linear transformations (also called linear functions) from n-dimensional space to m-dimensional space, where m is usually equal to n, and that's often 2 or 3. A linear transformation $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ is one that preserves addition and scalar multiplication, that is, $\mathbf{f}(\mathbf{a}+\mathbf{b}) = \mathbf{f}(\mathbf{a})+\mathbf{f}(\mathbf{b})$, and $\mathbf{f}(c\mathbf{a}) = c\mathbf{f}(\mathbf{a})$. We'll generally use bold face for vectors and for vector-valued functions. So, here, $\mathbf{a}$ and $\mathbf{b}$ are vectors, that is, points, in $\mathbf{R}^n$, while c is a scalar, that is, a real number in $\mathbf{R}$, and $\mathbf{f}$ is a vector-valued function.

In multivariate calculus, we'll look at nonlinear functions $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ from n-dimensional space to m-dimensional space. We'll restrict ourselves, however, to those that can be differentiated, or at least continuous, just about everywhere.

In order to study these nonlinear functions, we'll use the linear functions from linear algebra. Thus, you'll need to know quite a bit about linear algebra.

Topics for review. Here's a list of the topics we'll review. The text covers them all in detail, and we'll look at nearly all of them, but concentrate on the most important ones.

We'll use most of these concepts as early as chapter 2, the chapter on derivatives of multivariate functions. A few we won't use until chapter 3, but we'll need all of them by the end of the course.

Section 1.1. Vectors in two and three dimensions. The standard (x, y) coordinate plane, and the standard (x, y, z) coordinate three-space. Notations for vectors. Geometric interpretation as vectors as displacements as well as vectors as points. Vector addition, the zero vector $\mathbf{0}$, vector subtraction, scalar multiplication, their properties, and their geometric interpretation.

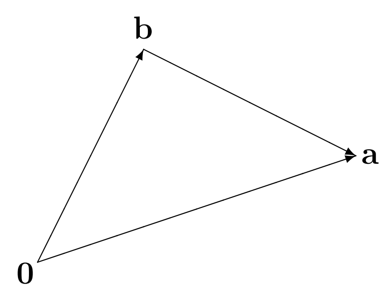
We start using these concepts right away. In chapter 2, we'll begin our study of vector-valued functions, and we'll use the coordinates, vector notation, and the rest of the topics reviewed in this section.

Section 1.2. Vectors and equations in $\mathbf{R}^3$. Standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbf{R}^3$. Parametric equations for lines. Symmetric form equations for a line in $\mathbf{R}^3$. Parametric equations for curves $\mathbf{x} : \mathbf{R} \to \mathbf{R}^2$ in the plane, where $\mathbf{x}(t) = (x(t), y(t))$.

We'll use standard basis vectors throughout the course, beginning with chapter 2. We'll study tangent lines of curves and tangent planes of surfaces in that chapter and throughout the course.

Section 1.3. The length $\|\mathbf{a}\|$ of a vector $\mathbf{a}$ (also called its norm). The triangle inequality.

$$\|\mathbf{a} - \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$$



Dot products. Dot products $\mathbf{a} \cdot \mathbf{b}$ of vectors (sometimes called inner products and denoted $\langle \mathbf{a}|\rangle\mathbf{b}$). Law of cosines and angles. Dot product of two vectors in terms of the angle between them

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta.$$

Two vectors are orthogonal, $\mathbf{a} \perp \mathbf{b}$, if and only if their dot product is 0. The projection $\text{proj}_{\mathbf{a}}\mathbf{b}$ of the vector **b** onto the vector **a** is

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|a\|^2}\right)\mathbf{a}.$$

Unit vectors. Normalization of vectors.

The concept of length of a vector is very important, especially in the form $\|\mathbf{b}-\mathbf{a}\|$ which gives the distance between two points **b** and **a**. In chapter 2 we'll define the limit of a function $\mathbf{f}:\mathbf{R}^n\to\mathbf{R}^m$ using this concept. In particular, we'll define

$$\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{f}(\mathbf{x}) = \mathbf{L}$$

as saying that we can make $\|\mathbf{f}(\mathbf{x})-\mathbf{L}\|$ arbitrarily small (less than ϵ) by keeping $\|\mathbf{x}-\mathbf{a}\|$ sufficiently small (less than δ).

Later in chapter 2 we'll see how directional derivatives can be found in terms of dot products, and we'll find tangent planes (and hyperplanes) in terms of dot products. We'll frequently use length and dot products throughout the course.

In one section of chapter 3 we'll study arc length and curvature, and in that section we'll see how unit vectors and normalization can be used to simplify their analysis.

Section 1.4. Cross products. Cross products $\mathbf{a}\times\mathbf{b}$ of pairs of vectors in $\mathbf{R}^3$. Areas of parallelograms and triangles in terms of cross products. Matrices and determinants as related to cross products. The triple scalar product, volume of a parallelepiped. Rotation and angular velocity.

Cross products and dot products together allow us to study geometry in three dimensions. In chapter 3 we'll look at Kepler's laws of planetary motion using derivatives of cross products and dot products of functions. Later in the chapter, we'll study the gradient, divergence, and curl of scalar and vector fields. These concepts depend on cross products and dot products.

Section 1.5. Equations and distance in $\mathbf{R}^3$. Equations of planes in three space including parametric equations of planes. Distance between a point and a line. Distance between parallel planes. Distance between skew lines.

Occasionally, we'll use these geometric concepts, especially in examples.

Section 1.6. Summary of geometry in n-space $\mathbf{R}^n$. The Cauchy inequality and the triangle inequality. The standard basis $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}$ for $\mathbf{R}^n$. $m\times n$ matrices as descriptions of linear functions $\mathbf{f}:\mathbf{R}^n\to\mathbf{R}^m$. Composition of these functions justifies matrix multiplication. Row and column vectors. Properties of matrix operations. Hyperplanes. Determinants, minors, and cofactors. A square matrix is invertible when its determinant is nonzero.

In chapter 2 we'll define the derivative of a function $\mathbf{f}:\mathbf{R}^n\to\mathbf{R}^m$ as the $m\times n$ matrix of partial derivatives. Later in that chapter, we'll see that the chain rule describes the derivative of a composition of two functions as the product of the two matrices describing the derivatives of the two functions.

Also in chapter 2 we'll see how determinants are used in the inverse and implicit function theorems.

We'll use matrices, vectors, and matrix products beginning in chapter 3 when we study the Hessian of a function. Determinants are also used in chapter 3 when we look at curl.

Section 1.7. Coordinate systems. Rectangular coordinates. Polar coordinates for planes. Conversion between rectangular and polar coordinates for the plane. Cylindrical and spherical coordinates for space. Conversion among the various coordinate systems for space.

We'll use these other coordinate systems in many of the examples we examine throughout the course.

Math 131 Home Page at
`http://math.clarku.edu/~djoyce/ma131/`