A Lorenz attractor. Math 131 Multivariate Calculus

D Joyce, Spring 2014

A Lorenz attractor. Sometimes surprising things can happen. Let's start with this vector field in space.

$$\mathbf{F}(x,y,z) = (10(y-x), 28x - y - xz, -\tfrac{8}{3}z + xy)$$

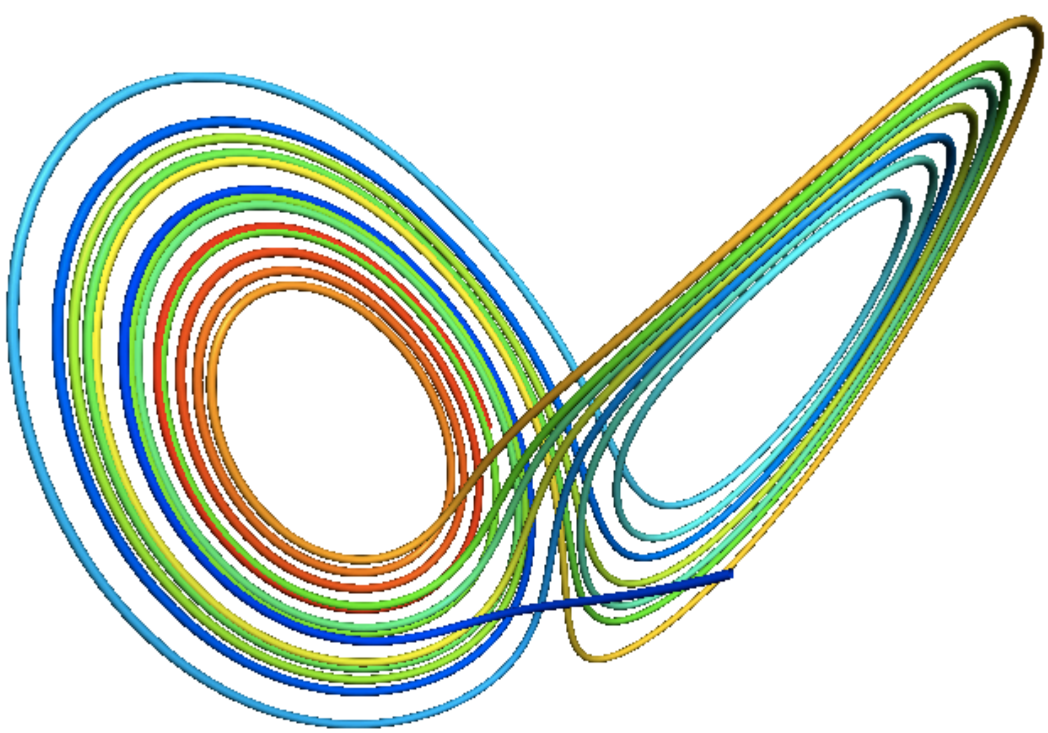
It seems pretty innocuous since it's just a polynomial vector field. We can, for instance, compute its divergence and curl.

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= -10 - 1 - \tfrac{8}{3} = -13.67 \end{aligned}$$

The divergence is a negative constant which suggests that there's some sort of uniform compression. It's this compression that makes it an attractor.

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &= (x - (-x), 0 - y, -z - 10) = (2x, -y, -z - 10) \end{aligned}$$

The curl isn't constant, but it's a linear polynomial in each coordinate. What could be simpler?



A flow line $(x, y, z)(t)$ for this vector field satisfies the system of linear differential equations

$$\begin{cases} x' &= 10(y-x) \\ y' &= 28x - y - xz \\ z' &= -\frac{8}{3}z + xy \end{cases}$$

As differential equations go, this is relatively straightforward. It's a system of three first order differential equations. The second and third equations aren't linear because of the terms xz and xy in them.

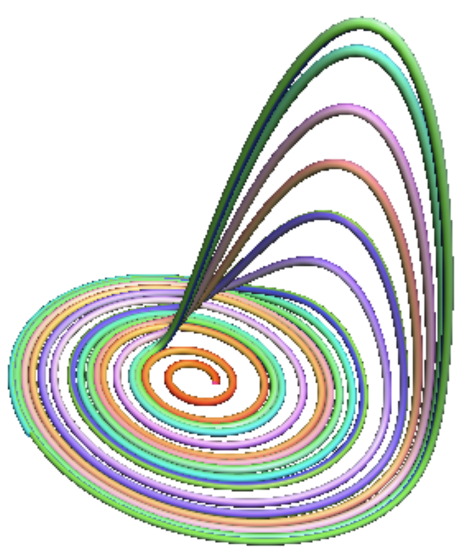
This is an example of a Lorenz attractor, a chaotic flow. One aspect of the chaos is that all nearby flow lines diverge. Figure 1 shows four images of the same flow from different perspectives. The initial point can be seen in some of the images.

The first image shows the two loops that the flow line runs around. The left loop flows clockwise while the right one flows counterclockwise. They two loops join together in the middle, the left loop coming in from the back and the right loop coming in from the front. As soon as they join they separate again, the left half of the flow going to the left loop, and the right half going to the right loop.

If you follow the flow line, it travels the left and right loops in what appears to be almost a random pattern. Somehow this deterministic system has the characteristics of a random system.

Rossler attractor. The Rossler attractor is another strange attractor. The one displayed here is given by the differential equations

$$\begin{cases} x' &= -y - z \\ y' &= x + 0.2y \\ z' &= 0.2 + xz - 5.7z \end{cases}$$



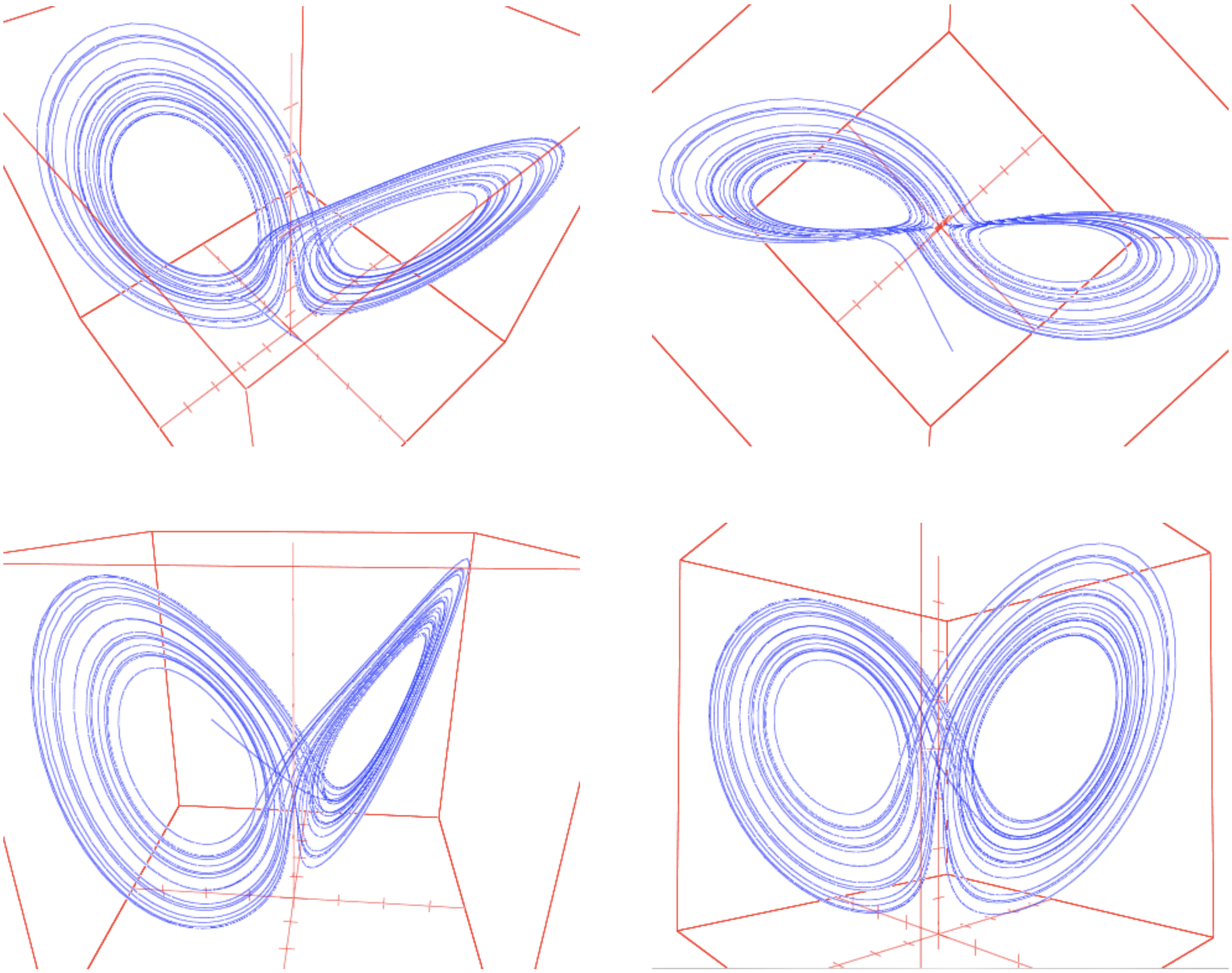


Figure 1: Four views of the Lorenz attractor

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