Partial derivatives
Math 131 Multivariate Calculus
D Joyce, Spring 2014

Last time. We looked at the definition of the multivariate limit $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$ as

$$\forall\epsilon > 0, \exists\delta > 0, \forall\mathbf{x} \in \mathbf{R}^n,$$
$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \epsilon;$$

some topological concepts; properties of limits; continuous functions; polynomials of several variables; component functions.

The derivatives you know. Recall from calculus that if $f : \mathbf{R} \to \mathbf{R}$ is a scalar-valued function of one variable, usually denoted $y = f(x)$, then its derivative at a scalar x, variously denoted $f'(x), y', \dfrac{df(x)}{dx}$, or $\dfrac{dy}{dx}$, is defined in terms of limits as

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

All the usual rules of differentiation followed from this definition.

Partial derivatives of scalar fields. We'll begin our generalization of derivatives by considering a scalar-valued function of several variables, $f : \mathbf{R}^n \to \mathbf{R}$, also called *scalar fields* on $\mathbf{R}^n$. Although $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ depends on n different variables, we can concentrate on what happens when we change just one of them by leaving all the other variables fixed. That leads to the concept of "partial derivative." Let's take $n = 3$ so we don't have to deal with subscripts, and then $f(\mathbf{x}) = f(x, y, z)$. Let's let x be the variable that we allow to change, and fix the other two variables y and z. Then we've converted f to a function of just one variable, and we already have a concept of derivative, namely,

$$\lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}.$$

Note how y and z don't change in this limit.

This kind of derivative is called a *partial derivative* since only one of the variables changes. Furthermore, the concept of limit used here is just the scalar limit you used in calculus, not the vector limit we discussed at the last meeting. There are various notations for this limit, the most common being a variation of Leibniz' notation, specifically $\dfrac{\partial f}{\partial x}$. The symbol ∂, which is read "partial" is a variant of the letter d, and it's only used to emphasize that there are other partial derivatives besides the one with respect to x. Thus,

$$\frac{\partial f}{\partial x}(x, y, z) = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h},$$
$$\frac{\partial f}{\partial y}(x, y, z) = \lim_{h\to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h},$$
$$\frac{\partial f}{\partial z}(x, y, z) = \lim_{h\to 0} \frac{f(x, y, z+z) - f(x, y, z)}{h}.$$

Sometimes other notations for partial derivatives are encountered. For instance, $D_x f$ and f_x are sometimes used for the partial derivative $\dfrac{\partial f}{\partial x}$. Note that primes, as in f' can't be used because they don't indicate which variable is changing.

Example 1. Let

$$f(x, y, z) = xz \sin(3y + 4z).$$

When you find $\dfrac{\partial f}{\partial x}$, just treat y and z as constants. Since $xz \sin(3y + 4z)$ is just x times some constant, therefore its derivative is just that constant, that is,

$$\frac{\partial f}{\partial x} = z \sin(3y + 4z).$$

Finding $\dfrac{\partial f}{\partial y}$ is a little more work for this example since the y is more deeply embedded in the mathematical expression. In particular, the chain rule is needed.

$$\frac{\partial f}{\partial y} = xz \frac{\partial}{\partial y}\big(\sin(3y + 4z)\big) = xz\big(\cos(3y + 4z)\big)3.$$

Finally, finding $\dfrac{\partial f}{\partial z}$ is even more work since the z occurs twice in the expression. The function f is a product of two functions, namely xz and $\sin(3y+4z)$, so we start by using the product rule

$$\begin{aligned} &\frac{\partial f}{\partial z} \\ =\ & \left(\frac{\partial}{\partial z}xz\right)\sin(3y+4z) + xz\frac{\partial}{\partial z}\big(\sin(3y+4z)\big) \\ =\ & x\sin(3y+4z) + xz\big(\cos(3y+4z)\big)4 \end{aligned}$$

There's really nothing to computing partial derivatives; they're just ordinary derivatives where only one variable actually varies. You'll see lots more examples in the text and homework exercises.

Geometric interpretation of partial derivatives. Let's find the geometric interpretation of a function $f(x,y)$ of two variables. Our question is: $\dfrac{\partial f}{\partial x}(a,b)$ is a derivative, so it's the slope of some tangent line. What tangent line?

For such a function, its graph $z=f(x,y)$ is a surface in $\mathbf{R}^3$. The point above (a,b) on that surface is (a,b,c), where the height above the (x,y)-plane is the value $c=f(a,b)$ of the function at (a,b). Whatever the tangent line is, it passes through this point. The surface $z=f(x,y)$ doesn't have just a tangent line at (a,b,c), it has a whole tangent plane. But if we hold y fixed at the value b, that tangent plane intersects the plane $y=b$ in a tangent line, and the slope of that tangent line is $\dfrac{\partial f}{\partial x}(a,b)$.

Likewise the tangent plane intersects the plane $x=a$ in another tangent line, and the slope of that tangent line is $\dfrac{\partial f}{\partial y}(a,b)$.

The tangent plane. We now have enough information to find the tangent plane. It's a plane that passes through the point $(a,b,c)=(a,b,f(a,b))$, and we know the lines of intersection of the two planes $x=a$ and $y=b$. That's enough information to conclude that the plane tangent to the surface $z=f(x,y)$ at the point $(a,b,f(a,b))$ has the equation

$$z = f(a,b) + \left(\frac{\partial f}{\partial x}(a,b)\right)(x-a) + \left(\frac{\partial f}{\partial y}(a,b)\right)(y-b).$$

In a different notation for partial derivatives, this equation becomes

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

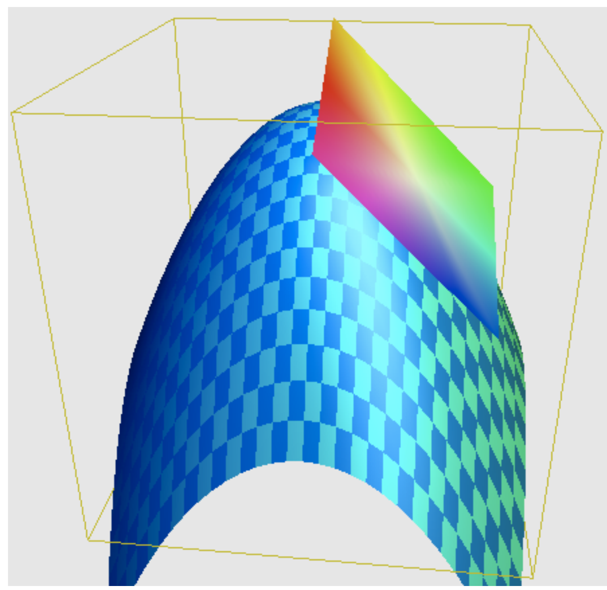

Figure 1: Tangent plane to $z=1-x^2-y^2$

Example 2. Let $f(x,y)=1-x^2-y^2$. The graph of this function is a paraboloid shown in figure 1. Let $(a,b)=(\frac14,\frac12)$. Then $f(\frac14,\frac12)=\frac{11}{16}$. We'll find the plane tangent to this surface at $(\frac14,\frac12,\frac{11}{16})$.

The partial derivatives of f are $f_x(x,y)=-2x$ and $f_y(x,y)=-2y$. Their values at $(a,b)=(\frac14,\frac12)$ are $f_x(\frac14,\frac12)=-\frac12$ and $f_y(\frac14,\frac12)=-1$. So the slope of the tangent plane in the x-direction is $-\frac12$, while the slope of the tangent plane in the y-direction is -1. Therefore, the equation of the tangent plane is

$$\begin{aligned} z &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= \tfrac{11}{16} - \tfrac12(x-\tfrac14) - (y-\tfrac12) \end{aligned}$$

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