Math 217/Econ 360

Final Exam Sample

Dec 2014

You may refer to two sheets of notes on this test, the summary table of distributions below, the table for the normal distribution, and you may use a calculator. Show your work for credit. You may leave your answers as expressions such as $\binom{8}{4}\frac{e^{1/3}}{\sqrt{2\pi}}$ if you like.

1. One of the cards of an ordinary deck of 52 cards is lost.

a. Given that it is not a king, what is the probability that it's an ace?

b. Given that it's either a heart or a diamond, what is the probability that it's an ace.

c. Is the event that the lost card is not a king independent of the event that the lost card is an ace?

d. Is the event that the lost card is either a heart or a diamond independent of the event that the lost card is an ace?

2. Let X be a continuous random variable with cumulative distribution function

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sqrt{x} & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 \le x \end{cases}$$

a. Determine the probability that $\frac{1}{3} \le X \le \frac{2}{3}$.

b. Determine the density function $f_X(x)$.

c. Determine the mean μ_X of X.

d. Determine the variance σ_X^2 of X.

3. Suppose that X is a random variable with moment generating function $g(t) = (0.3 + 0.7e^t)^5$. Note that the first and second derivatives of g are

$$g'(t) = 3.5(0.3 + 0.7e^t)^4 e^t, \text{ and } g'' = 3.5\big(0.3 + 3.5e^t\big)(0.3 + 0.7e^t)^3 e^t.$$

a. Determine the mean μ_X of X.

b. Determine the variance σ_X^2 of X.

4. Ann and Peter sell cars at a local dealership. The number of cars which Ann sells during a week is assumed to have a Poisson distribution with an average of 5 cars/week, and the number of cars which Peter sells during a week is assumed to have a Poisson distribution with an average of 4 cars/week. Assume that the numbers of cars/week that Ann and Peter sell are independent.

a. What is the probability that Ann sells exactly 5 cars in a given week?

b. What is the probability that Peter sells no cars in a given week?

c. What is the probability that together they sell exactly 10 cars in a given week?

5. Suppose that (a) 1 out of 10,000 of women at age forty have breast cancer. Further, suppose that (b) 99% of the women who do have breast cancer will get a positive mammogram, and (c) 1% of the women who do not have breast cancer will also get a positive mammogram. (d) If a woman in this age group gets a positive mammogram, how likely is it that she actually has breast cancer?

a-c. Let E be the event that a randomly chosen women at age forty has breast cancer, and let F be the event that a randomly chosen woman has a positive mammogram. Express each of the statements (a) through (c) in terms of probability or conditional probability.

d. Given statements (a-c) are correct, determine the probability for (d). (Show your work. You can leave your answer as an expression involving numbers or you can evaluate it decimally using a calculator.)

6. A fair die is rolled 10 times.

a. Calculate the expected sum of the 10 rolls.

b. Calculate the expected variance of the 10 rolls.

7. Prove that if X and Y are identically distributed but not necessarily independent, then $\mathrm{Cov}(X+Y, X-Y) = 0$.

8. (page 390, 8.2–8.3) From past experience, a professor knows that the test score of a student taking her final exam is a random variable X with mean $\mu = 75$ and variance $\sigma^2 = 5$.

a. Let n be the students taking the test, and let $\overline{X} = \frac{1}{n}\sum X_I$ be the average of the scores on their tests. Determine the mean $\mu_{\overline{X}}$ and the variance $\sigma^2_{\overline{X}}$ of $\overline{X}$.

b. By the Central limit theorem, if n is large, then the mean $\overline{X}$ is approximately normal. Use that to estimate the number of students that would have to take the exam to ensure a probability of at least 0.9 that the class average be within 5 of 75.

9. Short essay question, one or two paragraphs long. Explain what the prior and posterior probabilities are in Bayesian statistics.

Distribution	Type	Mass/density function $f(x)$	Mean μ	Variance σ^2
$\text{UNIFORM}(n)$	D	$1/n$, for $x = 1, 2, \ldots, n$	$(n+1)/2$	$(n^2-1)/12$
$\text{UNIFORM}(a,b)$	C	$\dfrac{1}{b-a}$, for $x \in [a,b]$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
$\text{BERNOULLI}(p)$	D	$f(0) = 1-p, f(1) = p$	p	$p(1-p)$
$\text{BINOMIAL}(n,p)$	D	$\binom{n}{x} p^x (1-p)^{n-x}$, for $x = 0, 1, \ldots, n$	np	npq
$\text{GEOMETRIC}(p)$	D	$q^{x-1}p$, for $x = 1, 2, \ldots$	$1/p$	$(1-p)/p^2$
$\text{NEGATIVEBINOMIAL}(p,r)$	D	$\binom{x-1}{r-1} p^r q^{x-r}$, for $x = r, r+1, \ldots$	r/p	$r(1-p)/p^2$
$\text{HYPERGEOMETRIC}(N,M,n)$	D	$\dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, for $x = 0, 1, \ldots, n$	np	$np(1-p)$
$\text{POISSON}(\lambda t)$	D	$\frac{1}{x!}(\lambda t)^x e^{-\lambda t}$, for $x = 0, 1, \ldots$	λt	λt
$\text{EXPONENTIAL}(\lambda)$	C	$\lambda e^{-\lambda x}$, for $x \in [0, \infty)$	$1/\lambda$	$1/\lambda^2$
$\text{GAMMA}(\lambda, r)$ $\text{GAMMA}(\alpha, \beta)$	C	$\dfrac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}$ $= \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$, for $x \in [0, \infty)$	r/λ $= \alpha\beta$	r/λ^2 $= a\beta^2$
$\text{BETA}(\alpha, \beta)$	C	$\dfrac{1}{\mathrm{B}(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, for $0 \le x \le 1$	$\dfrac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
$\text{NORMAL}(\mu, \sigma^2)$	C	$\dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$, for $x \in \mathbf{R}$	μ	σ^2
$\text{CHISQUARED}(\nu)$	C	$\dfrac{x^{\nu/2-1} e^{x/2}}{2^{\nu/2}\Gamma(\nu/2)}$, for $x \ge 0$	ν	2ν