Math 217/Econ 360
First Test Answers
Oct 2014

Scale. 87–100 A. 72–86 B. 55–71 C. Median 88.

1. [12] A special deck of 100 cards is numbered from 1 through 100. The deck is shuffled and three cards are dealt. Let X be the first card dealt, Y the second card dealt, and Z the third card dealt. What is the probabilty that $X < Y < Z$? Explain your reasoning.

There are $3! = 6$ permutations of three things giving 6 possible orders:

$X < Y < Z$	$X < Z < Y$
$Y < X < Z$	$Y < Z < X$
$Z < X < Y$	$Z < Y < X$

By symmetry they're all equally probable. Therefore $P(X < Y < Z) = \frac{1}{6}$.

There's also a much, much more complicated way of finding this probability. It involves counting all the ways that $1 \leq x < y < z \leq 100$.

2. [12] An urn contains five red marbles and three green marbles. Two of the eight marbles are chosen at random (without replacement). What is the probability that they are both red? Explain your reasoning.

This is an example of a hypergeometric distribution with parameters $N = 5+3 = 8$, $M = 5$, $n = 2$, and $x = 2$, so the probability is

$$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{x}} = \frac{\binom{5}{2}\binom{3}{0}}{\binom{8}{2}} = \frac{10}{28} = \frac{5}{14}$$

You can also find the answer using conditional probability. The probability that the first marble is red is $\frac{5}{8}$, and the probability that the second marble is red given that the first was red is $\frac{4}{7}$. Therefore, the probability that they're both red is $\frac{5}{8}\frac{4}{7} = \frac{5}{14}$.

3. [20] Let X be the number of heads on 5 tosses of a fair coin.

a. Fill in this table for the probability mass function $f(x)$ of X

This is a binomial distribution with parameters $n = 5$ and $p = \frac{1}{2}$, so $f(x) = \binom{n}{x} p^x q^n - x = \binom{5}{x}\frac{1}{32}$.

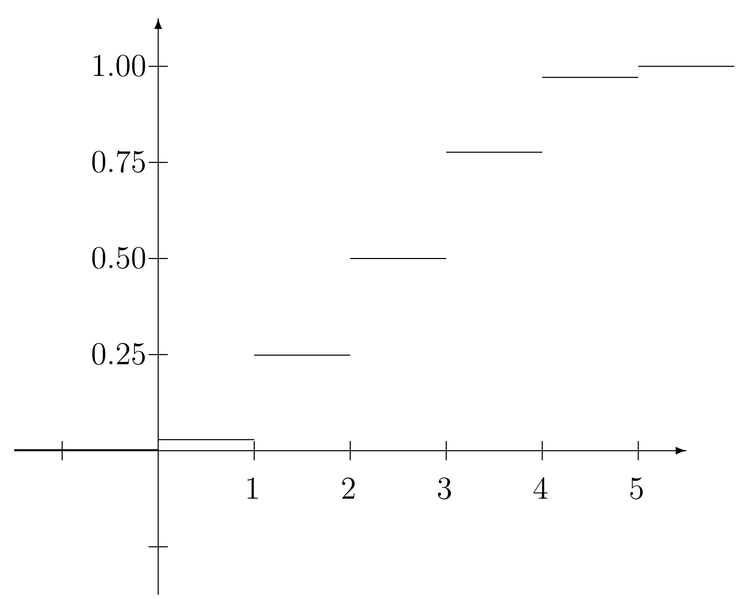
x	0	1	2	3	4	5
$f(x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

b. Draw the graph of the cumulative distribution function $F(x)$ of X

The cumulative distribution function $F(x)$ has a sum of values from $f(x)$. In tabular form it's

x	0	1	2	3	4	5
$F(x)$	$\frac{1}{32}$	$\frac{6}{32}$	$\frac{16}{32}$	$\frac{26}{32}$	$\frac{31}{32}$	1

Since this is a discrete distribution, it's c.d.f. is a step function. Its graph looks like



4. [12] Suppose that $P(A|B) = \frac{3}{5}$, $P(B) = \frac{2}{7}$, and $P(A) = \frac{1}{4}$. Determine $P(B|A)$.

Use Bayes' theorem.

$$\begin{aligned} P(A|B) &= \frac{P(A|B)\,P(A)}{P(B)} \\ &= \frac{\frac{3}{5}\frac{1}{4}}{\frac{2}{7}} = \frac{21}{40} \end{aligned}$$

5. [13] Prove that if A and B are independent events, then A and B^{c}, the the complement of B, are also independent events.

Since A and B are independent, $P(A \cap B) = P(A)\,P(B)$. Therefore

$$\begin{aligned} P(A)\,P(B^{\mathsf{c}}) &= P(A)\,(1 - P(B)) \\ &= P(A) - P(A)\,P(B)) \\ &= P(A) - P(A \cap B)) \\ &= P(A \cap B^{\mathsf{c}}) \end{aligned}$$

Thus, A and B^{c} are also independent.

6. [18] An experiment can result in one or both of the events A and B with these probabilities:

	A	A^{c}
B	0.34	0.46
B^{c}	0.15	0.05

Find the following probabilities:

a. $P(A) = P(A \cap B) + P(A \cap B^{\mathsf{c}}) = 0.34 + 0.15 = 0.49$.

b. $P(B) = P(A \cap B) + P(A^{\mathsf{c}} \cap B) = 0.34 + 0.46 = 0.80$.

c. $P(A \cap B) = 0.34$.

d. $P(A \cup B) = 0.34 + 0.46 + 0.15 = 0.95$.

e. $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.34}{0.80} = 0.425$.

f. $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{0.34}{0.49} = 0.69$.

7. [13] Prove Bayes formula: $P(F|E) = \dfrac{P(E|F)\,P(F)}{P(E)}$.

Any proof will have to use the definition of conditional probability twice.

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{P(E|F)\,P(F)}{P(E)}$$

The first equality is by the definition of $P(F|E)$, the second by the definition of $P(E|F)$.

Extra credit problems. For extra credit you may do the following problems. They are due Wednesday, Oct. 15. You may use the text and your notes, but don't get help from anyone. Work entirely by yourself. If you need any clarification on the problems, email me.

8. In order to win an election, a political candidate must win districts I, II, and III. Polls have shown that

the probability of winning both I and III is 0.55,
the probability of losing II but not I is 0.34, and
the probability of losing II and III but not I is 0.15.

Use these three probabilities to determine the probability that the candidate will win all three districts.

9. A fair die is tossed nine times. What is the probability of observing exactly two 3's, exactly three 1's, and exactly four 5's (and therefore no 2's, 4's, or 6's)?

10. An urn contains 8 green balls, 10 yellow balls, and 12 red balls. Six balls are removed from the urn (without replacement). Given that no green balls are chosen, determine the conditional probability that there are exactly 2 yellow balls among the chosen 6.