Expectation and variance for continuous random variables
Math 217 Probability and Statistics
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Today we'll look at expectation and variance for continuous random variables. We'll see most everything is the same for continuous random variables as for discrete random variables except integrals are used instead of summations.

Expectation for continuous random variables. Recall that for a discrete random variable X, the expectation, also called the expected value and the mean was defined as

$$\mu = E(X) = \sum_{x \in S} x\, P(X=x).$$

For a continuous random variable X, we now define the *expectation*, also called the *expected value* and the *mean* to be

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx,$$

where $f(x)$ is the probability density function for X. If $f(x)$ is 0 outside an interval $[a, b]$, then the integral is the same as

$$\mu = E(X) = \int_a^b x f(x)\, dx.$$

When $f(x)$ takes nonzero values on all of $\mathbf{R}$, then the limits of integration have to be $\int_{-\infty}^{\infty}$, and this is an improper integral. An improper integral of this form is defined as a sum of two improper integrals

$$\int_{-\infty}^{0} x f(x)\, dx + \int_0^{\infty} x f(x)\, dx,$$

and both have to be finite for the integral $\int_{-\infty}^{\infty}$ to exist. Improper integrals with infinite limits of integration can be evaluated by taking limits. For example,

$$\int_0^{\infty} x f(x)\, dx = \lim_{b\to\infty} \int_0^b x f(x)\, dx.$$

The value of the integral $\int_{-\infty}^{\infty} x f(x)\, dx$ can be interpreted as the x-coordinate of the center of gravity of the plane region between the x-axis and the curve $y = f(x)$. It is the point on the x-axis where that region will balance. You may have studied centers of gravity when you took calculus.

Properties of expectation for continuous random variables. They are the same as those for discrete random variables.

First of all, expectation is linear. If X and Y are two variables, independent or not, then

$$E(X+Y) = E(X) + E(Y).$$

If c is a constant, then

$$E(cX) = c\,E(X).$$

Linearity of expectation follows from linearity of integration.

Next, if Y is a function of X, $Y = \phi(X)$, then

$$E(Y) = E(\phi(X)) = \int_{-\infty}^{\infty} \phi(x) f(x)\, dx.$$

Next, if X and Y are independent random variables, then

$$E(XY) = E(X)\,E(Y).$$

The proof isn't hard, but it depends on some concepts we haven't discussed yet. I'll record it here and we'll look at it again after we've discussed joint distributions.

$$\begin{aligned}
E(XY) &= \int_{x\to-\infty}^{\infty}\int_{y\to-\infty}^{\infty} xy f_{XY}(x,y)\,dy\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y)\,dy\,dx \\
&= \int_{-\infty}^{\infty} x f_X(x) \int_{-\infty}^{\infty} y f_Y(y)\,dy\,dx \\
&= \int_{-\infty}^{\infty} y f_Y(y)\,dy \int_{-\infty}^{\infty} x f_X(x)\,dx \\
&= E(Y)\,E(X)
\end{aligned}$$

Variance and standard deviation for continuous random variables. When we discussed variance $\sigma^2 = \mathrm{Var}(X)$ for discrete random variables, it was defined in terms of expectation, so we can use the exact same definition and the same results hold.

$$\sigma^2 = \mathrm{Var}(X) = E((X-\mu)^2) = E(X^2) - \mu^2$$

$$\mathrm{Var}(cX) = c^2\,\mathrm{Var}(X)$$

$$\mathrm{Var}(X+c) = \mathrm{Var}(X)$$

If X and Y are independent random variables, then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

The proof of that last result depends on joint distributions, so we'll put it off until later.

Of course, standard deviation is still defined as the square root of variance.

The mean and variance of a uniform continuous random variable. We'll work this out as an example, the easiest one to do.

Let X be uniform on the interval $[a,b]$. Then $f(x) = \dfrac{1}{b-a}$ for $x \in [a,b]$.

The mean of X is

$$\begin{aligned}
\mu = E(X) &= \int_a^b x\frac{1}{b-a}\,dx \\
&= \frac{1}{2(b-a)}\,x^2\Big|_a^b \\
&= \frac{b^2-a^2}{2(b-a)} \\
&= \tfrac{1}{2}(a+b)
\end{aligned}$$

Thus, the mean is just where we expect it to be, right in the middle of the interval $[a,b]$.

For the variance of X, let's use the formula $\mathrm{Var}(X) = E(X^2) - \mu^2$, so we'll need to compute $E(X^2)$.

$$\begin{aligned}
E(X^2) &= \int_a^b x^2\frac{1}{b-a}\,dx \\
&= \frac{1}{3(b-a)}\,x^3\Big|_a^b \\
&= \frac{b^3-a^3}{3(b-a)} \\
&= \tfrac{1}{3}(a^2+ab+b^2)
\end{aligned}$$

Therefore, the variance is

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - \mu^2 \\
&= \tfrac{1}{3}(a^2+ab+b^2) - \tfrac{1}{4}(a+b)^2 \\
&= \tfrac{1}{12}(b-a)^2
\end{aligned}$$

Since the variance is $\sigma^2 = \frac{1}{12}(b-a)^2$, therefore the standard deviation is $\sigma = (b-a)/\sqrt{12}$.

The mean and variance of an exponential distribution. For a second example, let X be exponentially distributed with parameter λ so that the probability density function is $f(x) = \lambda e^{-\lambda x}$.

The mean is defined as

$$\mu = E(X) = \int_0^{\infty} x\lambda e^{-\lambda x}\,dx.$$

Using integration by parts or tables, you can show that

$$\int \lambda x e^{-\lambda x}\,dx = -xe^{-\lambda x} - \tfrac{1}{\lambda}e^{-\lambda x},$$

so, when we evaluate that from 0 to ∞, we get $(-0-0) - (-0-\frac{1}{\lambda}) = \frac{1}{\lambda}$. Thus, the mean is $\mu = \frac{1}{\lambda}$. Thus, the expected time to the next event in a Poisson process is the reciprocal of the rate of events.

Now for the variance σ^2. Let's compute $E(X^2)$. That's $E(X^2) = \int_0^{\infty} x^2\lambda e^{-\lambda x}\,dx$. Using integration by parts twice or tables, you can show that

$$\int \lambda x^2 e^{-\lambda x}\,dx = -x^2 e^{-\lambda x} - \tfrac{1}{\lambda} 2xe^{-\lambda x} - \tfrac{1}{\lambda^2} 2e^{-\lambda x},$$

and that evaluates from 0 to ∞ as $2/\lambda^2$. Therefore, the variance is

$$\sigma^2 = E(X^2) - \mu^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

The lack of a mean and variance for a Cauchy distribution. Not every distribution has a mean and variance. We'll see in a minute that the Cauchy distribution doesn't. There are also distributions that have means but not variances, or, you could say, their variances are infinite. The Central Limit Theorem will require that the distribution in question does have both a mean and variance.

Let's try to compute the mean of a Cauchy distribution and see what goes wrong. It's density is $f(x) = \dfrac{1}{\pi(1+x^2)}$ for $x \in \mathbf{R}$. So its mean should be

$$\mu = E(X) = \int_{-\infty}^{\infty} \frac{x\,dx}{\pi(1+x^2)}$$

In order for this improper integral to exist, we need both integrals $\int_{-\infty}^0$ and $\int_0^\infty$ to be finite. Let's look at the second integral.

$$\int_0^\infty \frac{x\,dx}{\pi(1+x^2)} = \frac{1}{2\pi} \log(1+x^2)\Big|_0^\infty = \infty$$

Similarly, the other integral, $\int_{-\infty}^0$, is $-\infty$. Since they're not both finite, the integral $\int_{-\infty}^\infty$ doesn't exist. In other words $\infty - \infty$ is not a number.

Thus, the Cauchy distribution has no mean. What this means in practice is that if you take a sample $x_1, x_2, \ldots, x_n$ from the Cauchy distribution, then the average $\overline{x}$ does not tend to a particular number. Instead, every so often you will get such a huge number, either positive or negative, that the average is overwhelmed by it.

A computation of its variance of a Cauchy distribution shows that's infinite, too.

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