Math 217 Probability and Statistics

Prof. D. Joyce, Clark University

Fri, 27 Sep 2007

First test. Monday, Oct 1.

Today. Review, Bayes' formula, independent events.

Bayes' formula for inverting conditional probabilities. Frequently, we want to know $P(F|E)$, but we know $P(E|F)$ instead. How do you invert these conditional probabilities? Bayes, long before the formal foundations of probability were invented, that is, before the concept of sample space, figured out a way.

Actually, it's easy to see what Bayes' formula has to be. Since $P(F \cap E)$ equals both $P(F|E)\,P(E)$ and $P(E|F)\,P(F)$, therefore

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E)}.$$

When applying this formula, frequently $P(E)$ has to be computed from conditional probabilities. Since $P(E) = P(E \cap F) + P(E \cap \tilde{F}) = P(E|F)\,P(F) + P(E|\tilde{F})\,P(\tilde{F})$, therefore

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E|F)\,P(F) + P(E|\tilde{F})\,P(\tilde{F})}.$$

Independent events. We've been using independence informally for a while now. We've taken the phrase "the events E and F are independent" it to mean that the knowledge that E has occurred has no effect on whether or not F will occur, and vice versa. Now that we have the concept of conditional probability, we can take that to mean that $P(F|E) = P(E)$ and $P(E|F) = P(E)$. A little algebra shows that each of those conditions is equivalent to $P(E \cap F) = P(E)\,P(F)$. So we'll take that as our definition of independent events.

Definition. Events E and F are said to be *independent* if

$$P(E \cap F) = P(E)\,P(F).$$

Note that E and F are independent if and only if E and $\tilde{F}$ are independent. That is, if the event E is independent of F, then it's also independent of its complement $\tilde{F}$.

Product spaces. The most important situation for independent events arises from taking the product two sample spaces Ω_1 and Ω_2. We create a new sample space, called the *product* or *Cartesian product*, $\Omega_1 \times \Omega_2$, whose outcomes are ordered pairs (ω_1, ω_2) of outcomes, $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. Probabilities are assigned to the events in this product sample space so that, for $A_1 \subseteq \Omega_1$ and $A_2 \subseteq \Omega_2$, the probability of a product event $A_1 \times A_2 = \{(\omega_1, \omega_2) \,|\, \omega_1 \in A_1 \text{ and } \omega_2 \in A_2\}$ is $P(A_1 \times A_2) = P(A_1)\,P(A_2)$.

Frequently, Ω_1 and Ω_2 are the same sample space Ω, so the product sample space $\Omega \times \Omega$ is denoted Ω^2.

Of course, products of more than two sample spaces are defined similarly.

We've seen many examples of product spaces before we had this definition. For example, if Ω is the 6-outcome sample space for a fair die, then Ω^2 is the 36-element sample space for a pair of dice. For another example, take the uniform continuous distribution on the unit interval $\Omega = [0, 1]$. Then the uniform continuous distribution on the unit square $[0, 1] \times [0, 1]$ is the product space.

Joint random variables. When we use the notation of joint random variables, we're implicitly using product spaces. For example, if X is the outcome for one fair die, and Y the outcome for another fair die, then (X, Y) is the joint random variable for the outcome of the pair of dice. It's a random variable on the product space Ω^2 having 36 elements described above.

When we have n independent trials X_1, X_2, $\ldots$, X_n of the same experiment, the joint random variable $\overline{X} = (X_1, X_2, \ldots, X_n)$ is a random variable on a product space.