Math 218 Mathematical Statistics

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Due Wednesday. Page 231: 17–20, 22.

Due Monday. Due: From chap. 7, page 262, exercises 1, 2, 9, 11, 12.

Last meeting. Discussed Type I and Type II errors on hypothesis tests, and α-risks and β-risks.

Today. We'll discuss 7.1.

z-confidence intervals and z-tests. Section 7.1. We consider the cases when (1) we have a normal distribution with a known variance σ^2, or (2) any distribution with a large sample. With a large sample, say $n \geq 30$, the central limit theorem says that $\overline{X}$ is approximately normal, and we can use the sample variance S^2 to approximate σ^2 if we don't happen to know σ^2.

In either case, the population mean μ is unknown, but it's estimated with the sample mean $\overline{X}$. This sample mean $\overline{X}$ is (at least almost) normally distributed with mean $\mu_{\overline{X}} = \mu$, variance $\sigma^2_{\overline{X}} = \sigma^2/n$, and standard deviation $\sigma_{\overline{X}} = \sigma/\sqrt{n}$. The normalized sample mean is $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$ if σ is known, and if σ is not known, for large n the sample standard deviation s very closely approximates σ, so we can use it instead. Since we know what the distribution of $\overline{X}$ is, we can use that knowledge to construct interval estimates for μ and hypothesis tests about μ.

z-intervals. The confidence intervals in this situation are called z-intervals because they're based on standard normal distributions and Z is the usual symbol for a standard normal random variable.

For historical reasons more than anything else, the confidence level is often taken to be 95%, so the associated level of significance is $\alpha = 0.05$. For a two-sided confidence interval, which is the usual kind, $\alpha/2$ of the area under each end of the standard normal density function is needed, and that means the cut-offs for the standard normal distribution are $\pm z_{\alpha/2}$. Here's a table of the these numbers for common confidence levels for one- and two-sided confidence intervals.

confidence	α	z_α	$\alpha/2$	$z_{\alpha/2}$
90%	0.1	1.28	0.05	1.645
95%	0.05	1.645	0.025	1.96
99%	0.01	2.33	0.005	2.575

You can find other values in table A.3 of the text. Note how you have to read the table backwards.

The two-sided z-interval at the α-level is

$$\left[\overline{x} - z_{\alpha/2}\,\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{\alpha/2}\,\frac{\sigma}{\sqrt{n}}\right],$$

while the lower one-sided confidence interval is

$$\left[\overline{x} - z_{\alpha}\,\frac{\sigma}{\sqrt{n}}, \infty\right),$$

and the upper one-sided confidence interval is

$$\left(-\infty, \overline{x} + z_{\alpha}\,\frac{\sigma}{\sqrt{n}}\right].$$

In the two-sided case, the *margin of error* is the quantity

$$E = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Typically, before an experiment is designed, both the α-level and the desired margin of error E are specified. From these, the size of the sample n can be determined. It is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E} \right)^2.$$

Hypothesis tests for μ. These are summarized in table 7.1 on page 240. They correspond to the three kinds of z-intervals. In the two-sided test, the null hypothesis is $H_0 : \mu = \mu_0$ where μ_0 is a constant. In the lower one-sided test the null hypothesis is $H_0 : \mu \geq \mu_0$, while in the upper one sided test $H_0 : \mu \leq \mu_0$.

The P-values for these three hypothesis tests for μ are summarized in table 7.2.

We'll look at example 7.2 in detail.