Fixed-Parameter Extrapolation and Aperiodic Order: Open Problems

Stephen Fenner*
University of South Carolina
fenner@cse.sc.edu

Frederic Green†
Clark University
fgreen@clarku.edu

Steven Homer‡
Boston University
homer@cs.bu.edu

August 13, 2018

1 Background

For a number of years we have been investigating the geometric and algebraic properties of a family of discrete sets of points in Euclidean space generated by a simple binary operation: pairwise affine combination by a fixed parameter, which we call *fixed-parameter extrapolation*. By varying the parameter and the set of initial points, a large variety of point sets emerge. To our surprise, many of these sets display *aperiodic order* and share properties of so-called "quasicrystals" or "quasilattices." Such sets display some ordered crystal-like properties (e.g., generation by a regular set of local rules, such as a finite set of tiles, and possessing a kind of repetitivity), but are "aperiodic" in the sense that they have no translational symmetry. The most famous of such systems are Penrose's aperiodic tilings of the plane [16]. Mathematically, a widely accepted way of capturing the idea of aperiodic order is via the notion of *Meyer sets*, which we define later. Our goal is to classify the sets generated by fixed-parameter extrapolation in terms of a number of these properties, including but not limited to aperiodicity, uniform discreteness, and relative density. We also seek to determine exactly which parameter values lead to which types of sets.

In this column, we set down the basic definitions, and highlight some of the main results obtained so far. Results are stated without proof, and we present just enough to convey an overall sense of the work, towards the goal of formulating some of our many open problems. For all details and proofs, we refer the reader to our paper at `https://arxiv.org/abs/1212.2889`. This work falls mainly in the area of discrete geometry, but we present several computational questions as well.

We begin with our main definition, which generalizes the notion of convexity. We usually work with the complex numbers $\mathbb{C}$, although we occasionally consider other fields F such as $\mathbb{R}$. In the standard definition, a set $A \subseteq \mathbb{C}$ is said to be convex iff, for any $a, b \in A$, for any real number $\lambda \in [0,1]$, we have $(1-\lambda)a + \lambda b \in A$. It is not hard to see that the definition still holds if we *fix* the parameter λ to be some real number strictly between 0 and 1, provided we take the (topological) closure of the resulting set. E.g., in the case of $\lambda = 1/2$, we say that if a topologically closed

*Computer Science and Engineering Department, Columbia, SC 29208. Partially supported by NSF grant CCF-0915948.

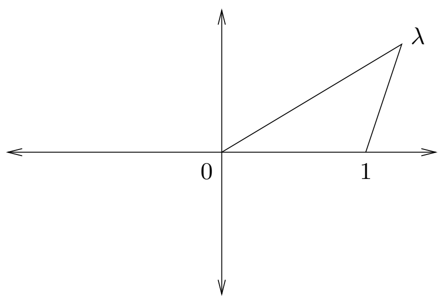
†Department of Mathematics and Computer Science, Clark University, Worcester, MA 01610

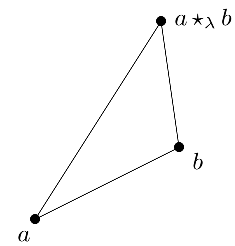
‡Computer Science Department, Boston University, Boston, MA 02215. Partially supported by NSF grant CCF-1533663.

set A is "closed under taking midpoints," then it is convex; the converse also holds. Thus in our main definition (which follows), we consider the case in which we fix λ (hence *fixed parameter*). However, we go a step further and relax the condition that λ is between 0 and 1, or even real (hence *extrapolation*).

Definition 1.1. *(Fixed-parameter extrapolation)*: Fix a number $\lambda \in \mathbb{C}$. For any $a, b \in \mathbb{C}$ define $a \star_\lambda b := (1-\lambda)a + \lambda b$. Then for any set $A \subseteq \mathbb{C}$, we say that A is λ-*convex* iff for every $a, b \in A$, the point $a \star_\lambda b$ is in A. We say that A is *nontrivial* if A contains at least two distinct elements.

Thus, as explained above, by the usual definition of convexity, a set $A \subseteq \mathbb{C}$ is convex if and only if, for all $\lambda \in (0,1)$, A is λ-convex. Also, if A is a (topologically) *closed* set, then for *any fixed* $\lambda \in (0,1)$, we have that A is convex if and only if A is λ-convex. We are generally interested in λ-convexity for $\lambda \notin [0,1]$, especially $\lambda \notin \mathbb{R}$, and we are particularly interested in minimal nontrivial λ-convex sets. Here is what the $\star_\lambda$ operation looks like geometrically, for a typical nonreal λ:





The two triangles are similar and oriented the same way.

Definition 1.2. For any $\lambda \in \mathbb{C}$ and any set $S \subseteq \mathbb{C}$, we define the λ-*closure* of S, denoted $Q_\lambda(S)$, to be the $\subseteq$-minimum λ-convex superset of S. We let Q_λ be shorthand for $Q_\lambda(\{0,1\})$, the λ-closure of $\{0,1\}$.

Q_λ is a minimal nontrivial λ-convex set because it is generated by just two distinct points. Furthermore, properties of sets $Q_\lambda(S)$ for $|S| > 2$ can often be inferred from Q_λ. The points 0 and 1 are often chosen for convenience as well, but since λ-convexity is invariant under $\mathbb{C}$-affine transformations, any two initial points would yield a set with the same essential properties.

After obtaining the results we discuss here, we discovered that fixed-parameter extrapolation (under various names, including, for example, s-convexity, quasicrystal addition, quasiaddition, and τ-inflation) had previously been investigated in the literature, and has quite a long history, especially in the context of the study of quasicrystals. The earliest to investigate the operation were Calvert [3] and, a few years later, Pinch [17], who proved a number of properties of λ-convex subsets of $\mathbb{R}$ for real λ. Berman & Moody [2] were the first to apply the idea to the study of quasicrystals, specifically in the case $\lambda = 1 + \varphi$ (where φ is the golden ratio, encountered again below). This value of λ is significant, because it gives the simplest example where Q_λ is discrete and aperiodic. Subsequently, other authors have looked at other values of λ that result in quasicrystal structures (most relevant to our work are [9], [10], and [11]). However, as far as we know, all work up to this point has dealt exclusively with $\lambda \in \mathbb{R}$. Ours is the first in-depth study that considers non-real λ, introduces the idea of *strong PV numbers* (which generalize the subset of real irrationals considered in [17]), and furthermore, is applied to domains other than $\mathbb{R}$ and $\mathbb{C}$. Most of the generalizations to complex λ are far from trivial. We indicate, in the course of our discussion of specific results, how our work relates to or generalizes prior work.

1.1 Basic results

We begin this survey of our results with a few basic observations. First, owing to the continuity of the $\star_\lambda$ operation, one can easily show that if A is any λ-convex set, then its (topological) closure, denoted $\overline{A}$, is also λ-convex. Second, if $\overline{Q_\lambda}$ is convex (by the standard definition of convexity), then there are only three possibilities: It is the unit interval, the real line, or the entire complex plane.

Theorem 1.3. *Suppose $\overline{Q_\lambda}$ is convex.*

1. *If $\lambda \in [0,1]$, then $\overline{Q_\lambda} = [0,1]$.*
2. *If $\lambda \in \mathbb{R} \setminus [0,1]$, then $\overline{Q_\lambda} = \mathbb{R}$ (essentially proved in [17, Proposition 7]).*
3. *If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\overline{Q_\lambda} = \mathbb{C}$.*

In light of this theorem, most of our work concentrates on determining, for various $\lambda \in \mathbb{C}$, whether or not $\overline{Q_\lambda}$ is convex, and if not, characterizing Q_λ. To this end, we can begin by broadly categorizing Q_λ as either having convex closure ($\lambda \in \mathcal{C}$) or being discrete ($\lambda \in \mathcal{D}$).

Definition 1.4. Let $\mathcal{C} := \{\lambda \in \mathbb{C} : \overline{Q_\lambda} \text{ is convex}\}$. Let $\mathcal{D} := \mathbb{C} \setminus \mathcal{C}$.

Highlights of this characterization include the following:

1. There are several equivalent ways to characterize convexity for $\overline{Q_\lambda}$. It turns out that $\overline{Q_\lambda}$ is convex (i.e., $\lambda \in \mathcal{C}$), iff it is path-connected, iff it contains a path, iff it has an accumulation point, iff it contains two points less than a unit distance apart. (The accumulation point characterization was proved by Pinch for real λ [17, Proposition 7].) Thus either $\overline{Q_\lambda}$ is convex or else Q_λ is discrete (and $Q_\lambda = \overline{Q_\lambda}$).

2. So far to a limited extent, we can carve out regions in $\mathbb{C}$ where $\overline{Q_\lambda}$ is convex. E.g.,

   (a) $\overline{Q_\lambda}$ is convex for all $\lambda = x + iy$ where $0 < x < 1$ and $-1 \le y \le 1$, except for the two points $e^{i\pi/3}$ and $e^{-i\pi/3}$.

   (b) If $\mathrm{Re}(\lambda) = 1/2$ (so λ makes an isosceles triangle with 0 and 1) and $|\lambda| \le \sqrt{3}$, then $\overline{Q_\lambda}$ is convex if and only if $|\lambda| \notin \{1, \sqrt{2}, \sqrt{3}\}$. ($Q_\lambda$ is discrete if $|\lambda| = \sqrt{n}$ for any positive integer n, but we don't know if the converse holds beyond $\sqrt{3}$.)

3. $\mathcal{C}$ is a big set: It is open and includes all transcendental numbers. Hence $\mathcal{D}$ is countable. Furthermore, $\mathcal{D}$ has no accumulation points in $\mathbb{C}$. (This was proved by Pinch for real λ [17, Theorem 8, Propositions 9 & 11].)

4. $\mathcal{D}$ consists only of algebraic integers. (Again, proved by Pinch in the real case [17, Proposition 12].)

Thus our primary interest is in $\lambda \in \mathcal{D}$. The simplest case to consider is that in which λ is an integer. Even here, our characterization of which integers are in Q_λ is partial, and this easy-to-prove result leads to one of our conjectures (Conjecture 2.2).

Proposition 1.5. *For all $\lambda \in \mathbb{Z}$ such that $\lambda \ge 2$ and all $n \in Q_\lambda$, $n \equiv d \pmod{\lambda(\lambda-1)}$ for some $d \in \{0, 1, \lambda, 1-\lambda\}$. More compactly,*

$$Q_\lambda \subseteq \lambda(\lambda-1)\mathbb{Z} + \{0,1,\lambda,1-\lambda\} = (\lambda\mathbb{Z} + \{0,1\}) \cap ((\lambda-1)\mathbb{Z} + \{0,1\}) .$$

(The second equality is by the Chinese remainder theorem.)

The most intriguing results are for certain algebraic integers λ, especially those (e.g., $1 + \varphi$, where φ is the Golden Ratio) not belonging to any discrete subring of $\mathbb{C}$. In fact, all cases we currently know of where Q_λ is discrete follow from a single result (Theorem 1.9, given below). We require some further algebraic and geometric background to state that theorem and explain its significance.

1.2 sPV Numbers, Delone sets, and Meyer sets

A *Pisot-Vijayaraghavan number* (or *PV number* for short)[1] is an algebraic integer $\alpha > 1$ whose Galois conjugates α' (other than α) all lie inside the unit disk in $\mathbb{C}$, i.e., satisfy $|\alpha'| < 1$. PV numbers arise naturally in the study of aperiodic order. The notion can be relaxed to allow for non-real α by excluding both α and its complex conjugate α^* from the norm requirement. For our work, we need a stronger definition.

Definition 1.6. We call an algebraic integer $\alpha \in \mathbb{C}$ a *strong PV number* iff its conjugates, other than α and α^*, all lie in the open unit interval $(0, 1)$. In this case, we also say that α *is sPV*. We say that a strong PV number α is *trivial* if it has no conjugates other than α and α^* (i.e., no conjugates in $(0, 1)$). Otherwise, α is *nontrivial*.

Nontrivial strong PV numbers include $1 + \varphi$ and $2 + \sqrt{2}$, and there are infinitely many real, irrational—hence nontrivial—strong PV numbers. Every strong PV number greater than 1 is a PV number, but not conversely; for example, φ and $1 + \sqrt{2}$ are PV numbers but not sPV. Theorem 1.9 below implies that Q_λ is discrete for all strong PV numbers λ. This result does not extend to all PV numbers; for example, $\overline{Q_\varphi} = \overline{Q_{1+\sqrt{2}}} = \mathbb{R}$.

The geometric context for Theorem 1.9 derives from discrete geometry, particularly concepts relating to ordered but aperiodic point sets in Euclidean space. The theory of such sets gained much impetus following the experimental discovery of so-called "quasicrystals"—materials whose atomic arrangements share many properties of crystals (e.g., sharp spikes in X-ray diffraction patterns), but—unlike true crystals—lack translational symmetry. Much of the mathematical theory of the corresponding point sets came well before this experimental discovery, and is due to Yves Meyer [13, 14]. The theory also relates closely to aperiodic tilings of the plane [4, 5, 16]. This section draws somewhat from the recent exposition of Baake & Grimm [1] as well as papers by Moody [2, 15], and related papers by Lagarias [7, 8]. We need not present the concepts in their full generality.

Definition 1.7. Let X be any metric space (think $\mathbb{R}^n$), and let A be any subset of X with at least two elements.

- A is *uniformly discrete* iff there exists an $r > 0$ such that $|x - y| > r$ for all distinct $x, y \in A$.
- A is *relatively dense* (in X) iff there exists an $R > 0$ such that, for all $x \in X$ there exists $y \in A$ such that $|x - y| < R$.
- A is a *Delone set* (in X) iff it is both uniformly discrete and relatively dense (in X).

As we mentioned before, one can show that if Q_λ is discrete, then points in Q_λ are at least unit distance apart, thus Q_λ is uniformly discrete. This was shown by Pinch for real λ [17, Proposition 10], but the generalization to complex λ is *not* straightforward.

[1]These numbers are also called Pisot numbers in much of the literature.

One way to define point sets in Euclidean space with "aperiodic order" is via so-called "Meyer sets" (Definition 1.8, below), in particular those which lack any translational symmetry. It turns out that many Meyer sets are of the form $Q_\lambda(S)$ for certain λ and S. There are many equivalent characterizations of Meyer sets; see, for example, Moody [15, Theorem 9.1]. All Meyer sets are Delone sets, but Meyer sets have additional properties, such as finite local complexity, not shared by all Delone sets.[2]

One way of producing aperiodic Meyer sets is through a construction called a *cut-and-project scheme* (and this was Meyer's original approach). We do not define this concept here, except to say that it gives rise to the notion of a "model set'—also called a "cut-and-project set," a certain kind of Delone set—and Meyer sets are relatively dense subsets of these sets. (We give an example of a cut-and-project set below.)

Luckily for us, citing work of Meyer [13] and Lagarias [7] as well as his own work, Moody [15] characterizes Meyer sets in various ways, some of which do not require knowledge of cut-and-project schemes. We will take one of these characterizations as our "definition" of a Meyer set:

Definition 1.8 (Lagarias, Moody [7, 15])**.** Let A be a relatively dense subset of $\mathbb{R}^n$. Then A is a *Meyer set* if and only if $A - A$ is uniformly discrete, where $A - A = \{x - y \mid x, y \in A\}$ is the Minkowski difference of A with itself.

We can now state our main mechanism for obtaining discrete $Q_\lambda(S)$.

Theorem 1.9. *If λ is sPV, then $Q_\lambda(S)$ is uniformly discrete for any finite set $S \subseteq \mathbb{Q}(\lambda)$. In fact, $Q_\lambda(S) - Q_\lambda(S)$ is uniformly discrete for all such S.*

Remark. Note that we do not show that $Q_\lambda(S)$ is relatively dense. If true, then it would be a Meyer set. This is one of our primary open problems. See Open Question 2.5, below. □

Pinch proved the special case of Theorem 1.9 where $\lambda \in \mathbb{R}$ and $S = \{0, 1\}$ [17, Proposition 12]. Our proof of this theorem essentially shows that $Q_\lambda(S)$ is a subset of a model or cut-and-project set. Thus, for example, the right-hand side of Eq. (1) below is a cut-and-project set. Although it would then follow directly from previous results of Meyer [13] (also see [15]) that $Q_\lambda(S)$ is uniformly discrete, our paper includes a self-contained proof for the sake of completeness. (The case where λ is not real and $S = \{0, 1\}$ was proved by Rohit Gurjar [6].) Probably our most important conjecture is that the converse of this theorem holds (Conjecture 2.1).

To illustrate this with one specific case—where $S = \{0, 1\}$—the proof of Theorem 1.9 gives the following containment for all algebraic integers λ (letting d be the degree of λ):

$$Q_\lambda \subseteq \{p(\lambda) \mid p \in \mathbb{Z}[x] \;\;\&\;\; \deg(p) < d \;\;\&\;\; 0 \le p(\mu) \le 1 \text{ for all } \mu \in (0, 1) \text{ and conjugate to } \lambda\} \,. \quad (1)$$

When λ is sPV, then the right-hand side of (1) is uniformly discrete. As an example, equality holds in (1) for $\lambda = 1 + \varphi$, and we get that $Q_{1+\varphi} = \{1\} \cup \{\lceil n/\varphi \rceil + n\varphi \mid n \in \mathbb{Z}\}$. This set contains no infinite arithmetic progressions and hence has no translational symmetry. Except for 0 and 1, any two adjacent points in $Q_{1+\varphi}$ differ either by φ or by $1 + \varphi$.

The degree requirement on p can be dropped in (1), since any polynomial with degree $\ge d$ can be reduced modulo the minimal polynomial of λ without affecting its value on λ or any of the μ.

[2] Other properties of interest include repetivity, diffractivity, and aperiodicity. A Meyer set may or may not possess any of these additional properties. For definitions, see [1, 15].

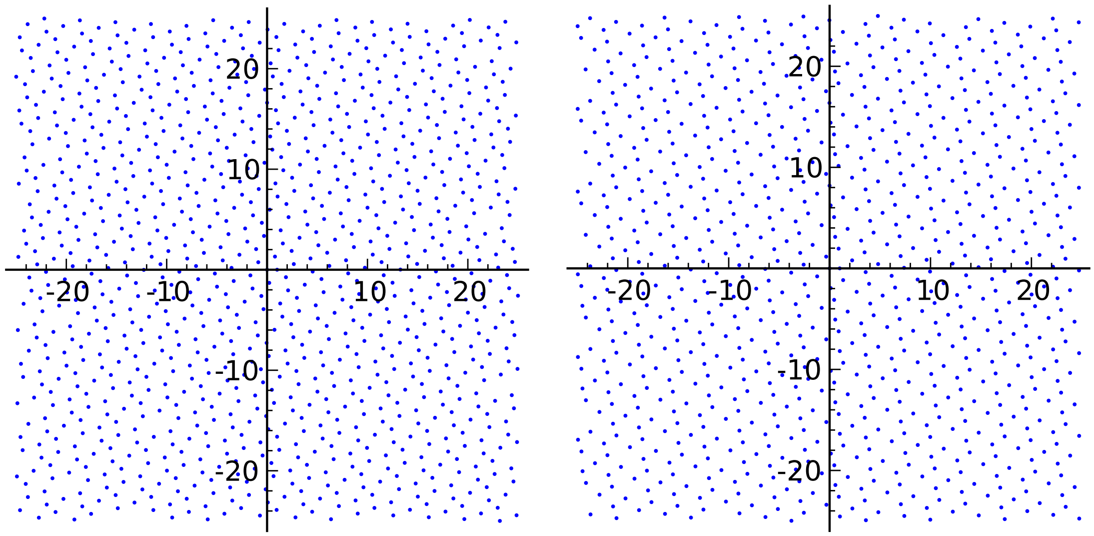


Figure 1: Plots of Q_λ for two nonreal values of λ. The left plot is of Q_λ, where λ is the root of the polynomial $x^3 + x^2 - 1$ closest to $-0.877 + 0.745i$. The right plot is of Q_μ, where μ is the root of the polynomial $x^3 - x^2 + 2x - 1$ closest to $-0.341 + 1.162i$. One can show that $Q_\lambda = Q_{\lambda^2} = Q_{\lambda^3}$.

A question intimately related to Conjecture 2.1, is this: For which sPV λ does equality hold in (1) (as well as other more general cases, e.g., for $S \neq \{0,1\}$)? To get a better sense of this question, compare the right-hand side of Eq. (1) with that of Eq. (2) below, which holds for *all* λ:

$$Q_\lambda = \{0,1\} \cup \{p(\lambda) \mid p \in \mathbb{Z}[x] \ \ \& \ \ 0 < p(\mu) < 1 \text{ for all } \mu \in (0,1)\ \} \ . \tag{2}$$

Figure 1 shows plots of Q_λ for two nonreal values of λ. The dots appear regularly spaced, but neither of these sets is periodic. We conjecture that equality holds in (1) for both these sets.

1.3 Higher Dimensions

It is interesting to study the λ-closure of regular polygons in the complex plane, where λ is real, but the points in S are complex. Letting P_n denote the regular n-gon and defining

$$\lambda_n := \frac{1}{2(1 - \cos(\pi/n))} \ ,$$

we have determined for certain n that the λ_n-closures of P_n and P_{2n} are aperiodic discrete sets. For example, Figure 2 displays a number of such sets. We include $Q_{1+\varphi}(P_5)$ (noting that $1+\varphi = \lambda_5$), as well as the λ_7-closures of P_7 (the 7-sided "heptagon") and P_{14}. The $(1+\varphi)$-closures of P_5 and P_{10} were also displayed by Berman & Moody [2], who concentrated their work on this particular value of λ. The values $\lambda_4 = 2 + \sqrt{2}$ and $\lambda_6 = 2 + \sqrt{3}$ are associated with 8- and 12-fold symmetry [10]. It is fairly easy to show, using the main theorem, that these sets are discrete. Our definition of λ_n, which is based on a geometric construction using P_n, yields other sets like these. These sets are discrete for only a finite number of odd values of n, namely $n = 3, 5, 7, 9, 15$, but are discrete for some even n as well. For example, Figure 2 includes the λ_6-closure of the duodecagon P_{12}.

We have also investigated similar constructions in three dimensions.

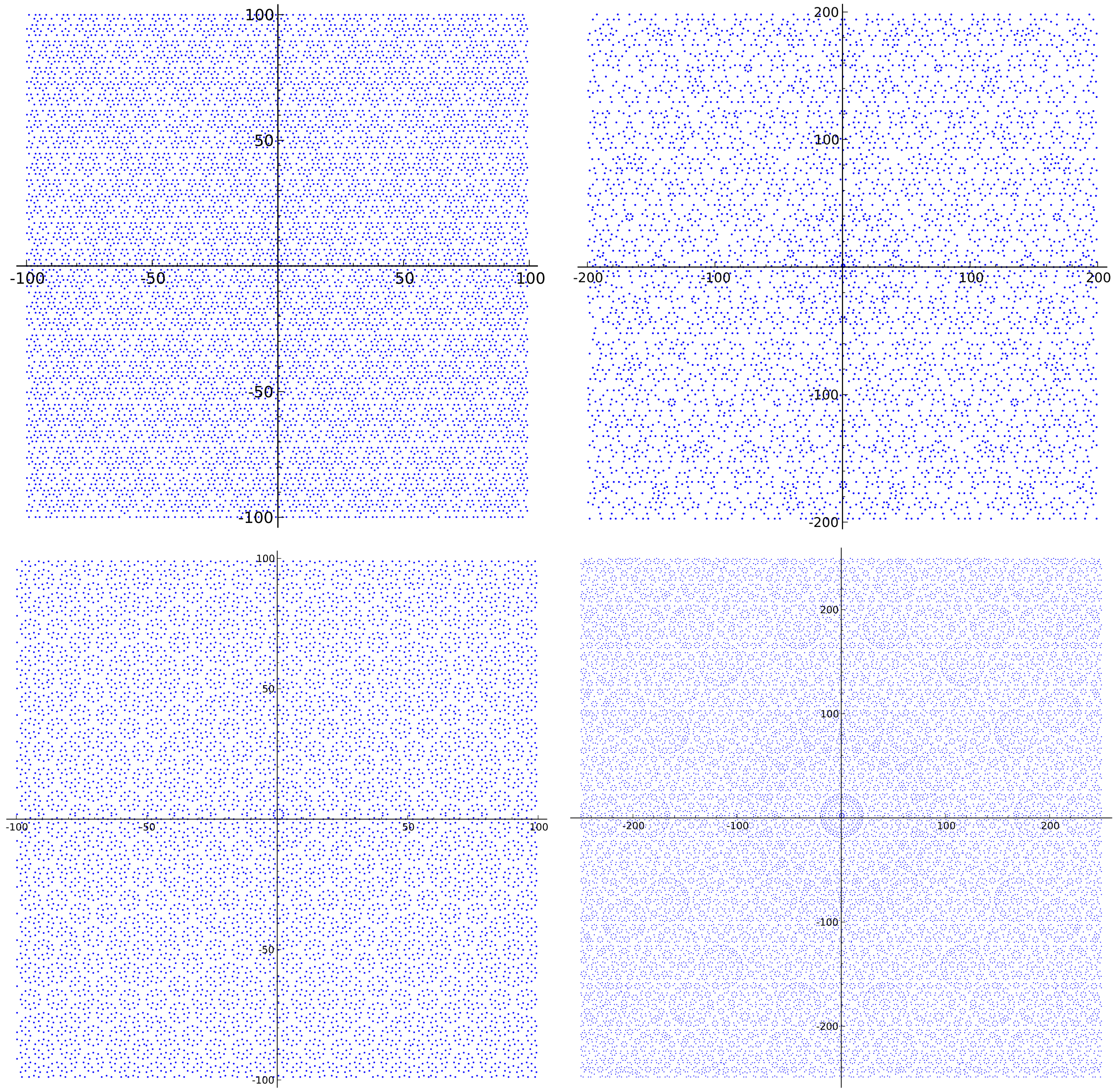


Figure 2: Top left: $Q_{1+\varphi}(P_5)$, the $(1+\varphi)$-closure of a regular pentagon. Top right: $Q_{\lambda_7}(P_7)$, the λ_7-closure of a regular heptagon. $\lambda_7 \approx 5.049$ and has minimal polynomial $x^3 - 6x^2 + 5x - 1$. Bottom left: $Q_{\lambda_6}(P_{12})$, the λ_6-closure of a regular 12-gon. Bottom right: $Q_{\lambda_7}(P_{14})$, the λ_7-closure of a regular 14-gon

2 Open Problems

We have many more questions than we can investigate in any reasonable length of time. We only give a sampling in this section. Some may be easy, but we have just not looked at them in depth.

Recall the set $\mathcal{C}$ (Definition 1.4) and its complement $\mathcal{D} := \mathbb{C} \setminus \mathcal{C}$. We know that $\mathcal{D}$ is closed, discrete, and contains only algebraic integers. We also know that $\mathcal{D}$ contains all strong PV numbers (Theorem 1.9), but we know of no other elements of $\mathcal{D}$ than these.

Conjecture 2.1. *Q_λ is discrete if and only if λ is a strong PV number.*

To make progress towards this conjecture, we can use various constructions we have developed to carve out more territory for $\mathcal{C}$ in the complex plane. This approach was started in our paper. But beyond the fact that $\mathcal{D}$ consists only of algebraic integers, we know little about $\mathcal{C}$ and $\mathcal{D}$.

There are a number of open questions about $\overline{Q_\lambda}$ when λ belongs to a discrete subring of $\mathcal{D}$. For example, we conjecture that equality holds in Proposition 1.5.

Conjecture 2.2. *For all $\lambda \in \mathbb{Z}$ such that $\lambda \geq 2$,*

$$Q_\lambda = \lambda(\lambda - 1)\mathbb{Z} + \{0, 1, \lambda, 1 - \lambda\} . \tag{3}$$

Eq. (3) is true if and only if $\lambda(\lambda-1) \in Q_\lambda$, which remains to be proved in general. Membership has been verified by computer for all λ into the hundreds.

The next open question (and its generalizations) is one of the most interesting.

Open Question 2.3. *For which sPV λ does set equality hold in (1)?*

When equality holds, Q_λ is a model set and relative density of Q_λ follows immediately. We have proved that equality holds for $\lambda = 1 + \varphi$, as well as $\lambda = 2 + \sqrt{2}$ and $2 + \sqrt{3}$. Our proof technique depends on specific algebraic properties of λ and fails for (e.g.) $\lambda = (5 + \sqrt{13})/2$. In fact, Masáková et al. [10] prove more strongly, via a different proof, that $1 + \varphi, 2 + \sqrt{2}$, and $2 + \sqrt{3}$ are the *only* "quadratic unitary Pisot" numbers[3] that lead to model sets in this way. We extend this to show that (with the *possible* exception of $(-3 - \sqrt{17})/2$), these are the only *quadratic sPV* numbers (of which unitary Pisot numbers are a special case) that lead to model sets. Thus we know that equality (which implies that Q_λ is a model set) does *not* hold for the quadratic unitary Pisot number $\lambda = (5+\sqrt{13})/2$, and *may* not hold for $(-3-\sqrt{17})/2$. Indeed, an extensive computer run finding points in Q_λ fails to produce the value $2\lambda - 1 = 4 + \sqrt{13} \approx 7.60555$. We nevertheless conjecture that relative density holds; see Open Question 2.5 below. The resulting sets could be Meyer or Delone, but not model sets.

Research Plan 2.4. *Prove that $4 + \sqrt{13} \notin Q_{(5+\sqrt{13})/2}$.*

All this leads to a general open question:

Open Question 2.5. *For F being either $\mathbb{R}$ or $\mathbb{C}$, is Q_λ relatively dense in F for all $\lambda \in F \setminus [0, 1]$? (If not, then for which λ?)*

If true, then the discrete Q_λ are all Meyer sets. We conjecture that this is indeed true, but so far we can only prove relative density in a small number of special cases, as enumerated in the discussion following Open Question 2.3. Proving relative density apparently will require new techniques. There are some indications that it could be false, however: a computer plot of $Q_{\lambda_{15}}(P_6)$ reveals what looks like self-similar, hierarchical structures consisting of clusters of points separated from each other but which collect to form new clusters when viewed at larger and larger scales.

[3] These are quadratic PV numbers λ such that $\lambda\lambda' = \pm 1$, where λ' is the conjugate of λ.

2.1 Higher Dimensions

There are some very significant open questions regarding discrete sets generated by regular polygons. For example, we know that $Q_{1+\varphi}(P_5)$ is both uniformly discrete and relatively dense; it is a Delone set (originally proved in [2]). This is also the case for the λ_5-closure of the decagon P_{10}. But we know nothing about the relative density of sets generated by the heptagon and other larger polygons.

Open Question 2.6. *For all n such that λ_n is sPV, are the sets $Q_{\lambda_n}(P_n)$ relatively dense (and hence Delone)?*

Our technique for proving relative density for the three sPV values enumerated above amounts to showing that Q_λ is a model set. For quadratic unitary sPV numbers other than those, we know we won't get model sets by [10], although we still might get Delone or Meyer sets. The situation becomes much more complicated for higher degrees. For example, we have compiled compelling evidence that $Q_{\lambda_7}(P_7)$, the set depicted in the upper right of Figure 2, is relatively dense. This includes a framework for a very unwieldy (and hence incomplete) proof, which nevertheless enables us to perform substantial computation and formulate a precise conjecture. λ_7 is the sPV number that is the largest magnitude root of the polynomial $x^3 - 6x^2 + 5x - 1$. Denote the other roots by μ and ν (both in $(0,1)$). Let $\beta = \mu + \nu - \mu\nu$, and define $n(m) := \lfloor m/\lambda_7 \rfloor$ and $\ell(m) := \lceil \beta m \rceil + n(m)$ for $m \in \mathbb{Z}$. For any $m \in \mathbb{Z}$, write $p(m) := -n(m) + \ell(m) \cdot \lambda_7 - m \cdot (\lambda_7)^2$.

Conjecture 2.7. *For all m, $p(m) \in Q_{\lambda_7}$.*

This is sufficient for proving that $Q_{\lambda_7}(P_7)$ is relatively dense. The conjecture has been verified for m up to $1,000,000$. We note in this case that Eq. 1 is apparently a *proper* containment, and we also know that Q_{λ_7} is not contained in $\{p(m) \mid m \in \mathbb{Z}\}$.

2.2 L-convex, auto-convex, and λ-semiconvex sets

Definition 2.8. Let L and S be any subsets of $\mathbb{C}$.

- S is *L-convex* iff S is λ-convex for all $\lambda \in L$.
- Let $Q_L(S)$ be the least L-convex superset of S.
- S is *auto-convex* iff S is S-convex.

All the Q_λ sets are auto-convex by the following result:[4]

Proposition 2.9. *For any $\lambda, \mu \in \mathbb{C}$, if $\mu \in Q_\lambda$, then Q_λ is μ-convex, and consequently, $Q_\mu \subseteq Q_\lambda$.*

The same goes for all $\overline{Q_\lambda}$. We conjecture the converse.

Conjecture 2.10. *If $S \subseteq \mathbb{C}$ contains 0 and 1, is topologically closed, and is auto-convex, then $S = \overline{Q_\lambda}$ for some $\lambda \in \mathbb{C}$.*

[4] Pinch proved a more general result for real λ: If $S \subseteq \mathbb{R}$ is auto-convex, then $Q_\lambda(S)$ is auto-convex [17]. The proof generalizes trivially to the complex numbers.

This conjecture implies that the set $\{\overline{Q_\lambda} \mid \lambda \in \mathbb{C}\}$ is closed under arbitrary intersections, because the intersection of any family of closed, auto-convex sets is clearly auto-convex (and closed).

Definition 2.11. Fixing $\lambda \in \mathbb{C}$, we will say that a set $S \subseteq \mathbb{C}$ is λ-*semiconvex* iff, for every $a, b \in S$, at least one of the points $a \star_\lambda b$ and $b \star_\lambda a$ is in S.

Clearly, if S is λ-semiconvex, then S is also $(1-\lambda)$-semiconvex. Note that the intersection of two λ-semiconvex sets need not be λ-semiconvex.

Research Plan 2.12. *For which $\lambda \in \mathbb{C}$ do there exist nontrivial, bounded λ-semiconvex sets?*

It is not too hard to see that if $|\lambda - 1/2| \leq 1/2$, then any disk (closed or open) is λ-semiconvex. There are at least two values of λ outside this range where bounded λ-semiconvex sets exist: if $\lambda = (1 \pm i\sqrt{3})/2$, then the vertices of any equilateral triangle form a λ-semiconvex set. What other such λ are there?

2.3 Computational Questions

Questions such as Research Plan 2.4 lead us to the following line of inquiry. If Q_λ is discrete, then each element of Q_λ can be expressed as a polynomial in λ, of fixed degree, with integer coefficients. Q_λ is countably infinite. Furthermore, it is computably enumerable: If $x \in Q_\lambda$, by enumerating all extrapolations beginning with $\{0, 1\}$, we will eventually obtain x. It is possible by similar reasoning that Q_λ is in NP; however it is not at all obvious that a succinct proof that $x \in Q_\lambda$ (to say nothing of $x \notin Q_\lambda$) can always be found, and its computational complexity is wide open. For those λ that satisfy Open Question 2.3, we have an effective procedure to determine if any x is in Q_λ. For other λ, however, especially if λ is not a strong PV number but Q_λ is nevertheless discrete, we do not know if Q_λ is decidable (although we do know that $Q_{[x]}$, defined in the next section, is decidable).

Open Question 2.13. *If Q_λ is discrete, is it decidable? Uniformly in λ? If it is decidable, what is its computational complexity? When is it true that $Q_\lambda \in \mathsf{NP}$, and how does this depend on λ?*

Note that if Conjecture 2.1 is false, it's possible that the answer depends on whether or not λ is a strong PV number. We conjecture that Q_λ is always decidable, but have no intuition regarding its containment in NP, to say nothing of P.

2.4 Miscellaneous Open Problems

It would be interesting to pin down $\mathcal{C} \cap \mathbb{R}$ and $\mathcal{C} \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) = 1/2\}$. These two cases may be easier than the general case, as they present symmetries not shared by all λ.

Research Plan 2.14. *Determine which $\lambda > 3$ yield $\overline{Q_\lambda} = \mathbb{R}$. Determine which λ with $\mathrm{Re}(\lambda) = 1/2$ yield $\overline{Q_\lambda} = \mathbb{C}$.*

Research Plan 2.15. *Get a reasonably good graphical picture of $\mathcal{D}$.*

By $Q_{[x]}$ we denote the set of polynomials in $\mathbb{Z}[x]$ generated by the constant polynomials 0 and 1, and by repeated applications of $\star_x$; that is, $Q_{[x]}$ is the smallest set of polynomials containing $0, 1$ and closed under the binary operation $(p, q) \mapsto (1-x)p + xq$.

$Q_{[x]}$ has some interesting properties. Any element of Q_λ can be written as $p(\lambda)$ where $p \in Q_{[x]}$, and conversely. Pinch showed [17, Corollary 4.1] that an integer polynomial p is in $Q_{[x]}$ if and only if there exist $n \geq 0$ and integers $b_0, \ldots, b_n$ such that $p(x) = \sum_{i=0}^{n} b_i x^i (1-x)^{n-i}$ and $0 \leq b_i \leq \binom{n}{i}$ for all $0 \leq i \leq n$. We have an alternate characterization of $Q_{[x]}$: an integer polynomial p is in $Q_{[x]}$ if and only if either $p \in \{0, 1\}$ or $0 < p(\mu) < 1$ for all $0 < \mu < 1$. This latter characterization can be used to computably enumerate the integer polynomials *not* in $Q_{[x]}$, leading to a decision procedure for $Q_{[x]}$. What, then, is the complexity of deciding $Q_{[x]}$?

There are other interesting (though noncomputational) questions regarding $Q_{[x]}$. We can list all 14 polynomials in $Q_{[x]}$ of degree ≤ 2, and get a finite upper bound on the number of polynomials in $Q_{[x]}$ of any given degree bound. However, we don't even know how many polynomials there are in $Q_{[x]}$ of degree 3.

Open Question 2.16. *How many elements of $Q_{[x]}$ are there of degree 3?*

Our techniques give an upper bound of 717, and an extensive computer search finds only 90. Perhaps the following fact can reduce the upper bound:

Fact 2.17. *Let $p \in \mathbb{R}[x]$ be any polynomial in $\mathbb{R}[x]$ such that $\{p(0), p(1)\} \subseteq \{0, 1\}$. Then $0 < p(\mu) < 1$ for all $0 < \mu < 1$ if and only if $0 < p(r) < 1$ for every root r of p' (the derivative of p) such that $0 < r < 1$.*

Other open problems include the following:

Call the triangle with vertices $(0, 1, \lambda)$ the *fundamental triangle*. George McNulty offers the following conjecture [12]:

Conjecture 2.18 (McNulty)**.** *If Q_λ contains a point in the interior of the fundamental triangle, then $\overline{Q_\lambda}$ is convex.*

Definition 2.19. We will call a discrete set Q_λ *maximal* if it is not a proper subset of any other discrete Q_μ.

Open Question 2.20. *Do maximal Q_λ exist? Is there an easy way to characterize the λ such that Q_λ is maximal? Is there an interesting notion of minimal Q_λ?*

Open Question 2.21. *Is there an easy way to characterize the minimum polynomials of strong PV numbers? Short of that, find such polynomials of higher and higher degree. (We currently can characterize all such polynomials of degree ≤ 4.)*

Open Question 2.22. *We say that a λ-convex set A is* cohesive *if $A = Q_\lambda(A \setminus T)$ for any finite set T. We can show that the right-hand side of (1) is cohesive for all unitary quadratic sPV λ—if you remove 0 and 1. This implies that if Research Plan 2.4 is true, then there are infinitely many points "missing" from $Q_{(5+\sqrt{13})/2}$. For which λ is $Q_\lambda \setminus \{0, 1\}$ cohesive? All sPV numbers, perhaps?*

Open Question 2.23. *Let A be λ-convex as in the previous question. An* essential point *of A is some $x \in A$ such that $A \setminus \{x\}$ is λ-convex*[5]*. For example, 0 and 1 are both essential points of Q_λ for all sPV λ. The question is: if $Q_\lambda \setminus \{0, 1\}$ is not cohesive, must it have an essential point? More generally, what is the smallest size of a set you can remove from a noncohesive set that leaves a λ-convex set?*

[5]And hence $x \notin Q_\lambda(A \setminus \{x\})$

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