AUTOMORPHISMS OF GRAPH GROUPS

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Abstract. Given a graph $\Gamma = (V, E)$, the *graph group on* Γ, F_Γ, is the group generated by the vertex set V with the defining relations being that two adjacent vertices commute. In the following, the centralizer problem for graph groups is solved and it is shown that, for some graph groups, $\mathrm{aut}(F_\Gamma)$ is finitely generated by generators which are natural analogues to the Neilsen automorphisms for a free group.

1. Introduction

Let $\Gamma = (V, E)$ be a graph. We define F_Γ, the *graph group on* Γ, be the group with presentation $\langle V \mid [E]\rangle$, where $[E]$ denotes the set of commutators $\{[a, b] : (a, b) \in E\}$. The class of graph groups contains as extreme cases the free and the free abelian groups. It is natural to consider those properties which are true for both free and free abelian groups and to inquire whether they remain true for graph groups. Similar objects were first studied by Kim and Roush, [6], and then by Kim, Makar-Limanov, Neggers, and Roush, [5], who were examining the *graph algebra*. Let K be a field, and Γ be a graph. The graph algebra, $K(\Gamma)$, is the K algebra generated by the vertices of Γ subject only to the relations that two adjacent vertices commute. Their main result was that two graph algebras are isomorphic if and only if their graphs are isomorphic. They also characterized those monomials in $K(\Gamma)$ which commute, leading to the theorem that the centralizer of a monomial in a graph algebra is also a graph algebra. Dicks, [1], found a free resolution of K as a $K(\Gamma)$ module with trivial action, which gives, for any ring R, an RF_Γ-free resolution of R. This resolution allowed Droms, [2], to compute the integral cohomology ring of F_Γ, which is what might be called the *graph exterior algebra*. Droms proved, following the methods of Kim et al, that two graph groups are isomorphic if and only if the graphs are isomorphic. This implies that the graphs are significant to the group structure and are not simply a convenience indescibing the presentation.

In the following we give necessary and sufficient conditions for two elements to commute in F_Γ. There are two obvious ways in which a pair of elements may commute in a graph group. On one hand, the elements may commute letter by letter, that is, they are represented by two words such that every letter occurring in the first word is adjacent in the graph to every letter occurring in the second word. On the other hand the two elements may belong to a cyclic subgroup of F_Γ. We show that a combination of these two effects accounts for all commutativity in F_Γ.

As an application, we examine the automorphism groups of graph groups. A striking fact about both free and free abelian groups is that their automorphism groups have essentially the same generators. If $A = A(X)$ is a free abelian group of rank N on the (ordered) set X, then $\text{aut}(A) = \text{GL}(N, Z)$, generated by the elementary matrices.

For a free group, $F(X)$, the elementary transformations written multiplicatively generate $\text{aut}(F(X))$ and are called the *Nielsen automorphisms*. They are defined on the generators of F as follows:

- **[N1]:** *Inversion:* ι_x for some $x \in X$ with $\iota_x(x) = x^{-1}$ and $\iota_x(z) = z$ for $z \in (X - x)$,
- **[N2]:** *Transposition:* $\sigma_{x,y}$ for $x, y \in X$, $\sigma_{x,y}(x) = y$, $\sigma_{x,y}(y) = x$ and $\sigma_{x,y}(z) = z$ for $z \in (X - \{x, y\})$,
- **[N3]:** *Transvection:* $\tau_{x,y}$ for $x, y \in X$, $\tau_{x,y}(x) = xy$, $\tau_{x,y}(z) = z$ for $z \in (X - x)$.

For a graph group, we define analogously the set of *elementary automorphisms*, of F_Γ to consist of the following

- **[EN1]:** *Inversion:* ι_x for some $x \in V$ with $\iota_x(x) = x^{-1}$ and $\iota_x(v) = v$ for $v \in (V - x)$,
- **[EN2]:** *Graphic Automorphism:* g_ϕ for $\phi \in \text{aut}(\Gamma)$, $g_\phi(v) = \phi(v)$ for all $v \in V$,
- **[EN3]:** *Dominated Transvection:* $\tau_{x,y}$, for $x, y \in V$ such that $(x, z) \in E \Rightarrow (y, z) \in E$, $\tau_{x,y}(x) = xy$, $\tau_{x,y}(v) = v$ for $v \in (V - x)$,
- **[EN4]:** *Locally Inner Automorphism:* $\lambda_{x,C}$, where $x \in V$ and C is a connected component of $\Gamma - \text{star}(x)$, with $L(v) = x^{-1}vx$ for $v \in C$, $L(v) = v$ for $v \in V - C$.

These reduce to the classical case when the graph is discrete or complete, and we will see that in the cases where Γ is a tree or if Γ is strongly two- connected with girth greater than four, then they do generate $\text{aut}(F_\Gamma$.

2. Words

Given a word w on the *letters* $V \cup V^{-1}$, the *length of* w, $|w|$, is the number of letters in the spelling of w, and the *support* of w, $\text{supp}(w)$, is the set of elements v in V such that v or v^{-1} occurs in the spelling of w.

Let $\Gamma = (V, E)$ be a graph, $F_\Gamma = \langle V \mid [E] \rangle$ the graph group associated with Γ, and let $u \in F_\Gamma$. Then we define the *length of* u, $|u|$, to be the least number of letters in any word representing u. We also define the *support of* u, $\text{supp}(u)$, to be the set of those vertices which occur in the support of every word representing u.

If w is a word containing no segment of the form vxv^{-1} or $v^{-1}xv$, where v is a vertex adjacent to every vertex in $\text{supp}(x)$, then we say that w is a *graphicly reduced word*. If w does contain such a segment, then this segment can be replaced by x, and it is easy to show that every word w can be transformed into a graphicly reduced word in at most $\frac{1}{2}|w|$ steps. It is easy to show that a given graphicly reduced word representing u can be transformed in to any other graphicly reduced word for u by a finite sequence of moves which do not affect the length of the words, i.e. replacing a segment ab by ba, where a and b are adjacent vertices, (or inverses of vertices). A word w is a graphicly reduced word for u if and only if $|w| = |u|$, and if w is a graphicly reduced word for u then $\text{supp}(u) = \text{supp}(w)$.

If $u, v \in F_\Gamma$ then $|uv| \le |u| + |v|$, and if there is equallity then we say that the product uv is a *reduced factorization*.

Let w be a graphicly reduced word and let (a, b) be pair of non-adjacent vertices. We define the *projection of w onto $\{a, b\}$*, denoted by $w_{\{a,b\}}$, to be the subword of w obtained by deleting those letters in w which are not one of a, a^{-1}, b, or b^{-1}. Of course, the projection of w onto (a, b) will be the empty word if $\mathrm{supp}(w)$ is disjoint from (a, b). Notice that the projection of a graphicly reduced word need not be graphicly reduced. For $u \in F_\Gamma$ we define the *projection of u onto (a, b)*, denoted by $u_{\{a,b\}}$, by $u_{\{a,b\}} = w_{\{a,b\}}$, where w is any graphicly reduced word representing u. It is easy to see that $u_{\{a,b\}}$ is well defined. We define $\mathrm{PB}(u)$, the *projection basis of u*, to be the set of all words which are projections of u onto pairs of non-adjacent vertices. In the same way, for any vertex a we define $u_{\{a\}}$ to be the projection of an element u onto a.

Lemma 1. *Let Γ be a graph whose complement Γ^c has no isolated vertices. Then if two elements of F_Γ have the same projection basis then they have the same length and support.*

Proof. Let u and w be elements in F_Γ and let v be a vertex of Γ. Then since v is not isolated in Γ^c there is a vertex v' which is non-adjacent to v. Since $u_{\{v,v'\}} = w_{\{v,v'\}}$, the number of occurrences of the vertex v in any graphicly reduced word for u or u' is determined, and the result follows. □

The following proposition amounts to a solution to the word problem.

Proposition 1. *Let Γ be a graph whose complement has no isolated vertices and let $u, v \in F_\Gamma$. Then $\mathrm{PB}(u) = \mathrm{PB}(v)$ if and only if $u = v$.*

Proof. We induct on $|u|$. If $|u| = |v| = 0$, then $u = v = 1$ and there is nothing to show. Now, let $|u| > 0$ and let a be a vertex in $\mathrm{supp}(u) = \mathrm{supp}(v)$ such that a or a^{-1}, say a, is the first letter in every projection in the projection basis $\mathrm{PB}(u) = \mathrm{PB}(v)$ in which it occurs. We know that such a vertex exists since u is represented by some graphicly reduced word, and the first letter in this word will have the property. Such a letter is not necessarily unique and will be called a *first letter of u*. We have then that $|a^{-1}u| = |a^{-1}v| = |u| - 1$. $\mathrm{PB}(a^{-1}u)$ is obtained from $\mathrm{PB}(u)$ by simply deleting a from the beginning of every projection in $\mathrm{PB}(u)$ in which it appears, and similarly for $\mathrm{PB}(a^{-1}v)$. So $\mathrm{PB}(a^{-1}u) = \mathrm{PB}(a^{-1}v)$, hence $a^{-1}u = a^{-1}v$ by induction and thus $u = v$. □

3. The Conjugacy Problem

Let u and v be elements of F_Γ. u and v are conjugate if and only if u can be transformed into v by a finite sequence of moves of the form $u \to aua^{-1}$, where a is a letter of F_Γ. It is clear that $|aua^{-1}|$ is equal to either $|u| - 2$, $|u|$, or $|u| + 2$, and it is easy to show that if u and v are conjugate then the sequence of moves can be chosen such that all length decreasing moves occur first.

We say that an element, u, in a graph group is *cyclicly reduced* if u cannot be written $u = au'a^{-1}$ with $|u| = |u'| + 2$. An element can be cyclicly reduced by a sequence consisting only of length reducing moves. The result does not depend on the particular sequence, which is the result of

Proposition 2. *Every element $u \in F_\Gamma$ is conjugate to a unique cyclicly reduced element,* $\mathrm{CR}(u)$*, $u = \alpha \mathrm{CR}(u) \alpha^{-1}$, such that $|u| = |\mathrm{CR}(u)| + 2|\alpha|$.*

Proof. Only uniqueness is at issue. Suppose there are cyclicly reduced element x and y such that $u = \delta x \delta^{-1} = \beta y \beta^{-1}$ are reduced factorizations of u. Let a be the first letter of δ. If a is not a first letter of β, then a is a first letter of y and a is not an element of $\mathrm{supp}(\beta)$. Hence a^{-1}, the last letter of δ^{-1}, must be a last letter of x, contradicting the fact that y is cyclicly reduced. So a is a first letter of both β and δ, and we have the reduced factorizations $\delta = a\delta'$, $\beta = a\beta'$, $\delta' x \delta'^{-1} = \beta' y \beta'^{-1}$, and the result follows by induction on $|u|$. □

Corollary 1. *If $u = pu'p^{-1}$, is a reduced factorization, then* $\mathrm{CR}(u) = \mathrm{CR}(u')$.

If u is conjugate to v then $|\mathrm{CR}(u)| = |\mathrm{CR}(v)|$. This reduces the conjugacy problem to that for cyclicly reduced elements of equal length, which is clear.

4. The Centralizer Problem

Let u and v be elements of F_Γ. We have that $|uv| \leq |u| + |v|$. There exists a unique h such that

$$\begin{aligned} u &= u'h \quad \text{with} \quad |u| = |u'| + |h|, \\ v &= h^{-1}v' \quad \text{with} \quad |v| = |v'| + |h|, \text{ and} \\ uv &= u'v' \quad \text{with} \quad |uv| = |u'| + |v'|. \end{aligned}$$

Let $\{a, b\}$ be a pair of non-adjacent vertices of Γ. The projections $u_{\{a,b\}}$ and $v_{\{a,b\}}$ satisfy $u_{\{a,b\}} = u'_{\{a,b\}} * h_{\{a,b\}}$, $v_{\{a,b\}} = h^{-1}_{\{a,b\}} * v'_{\{a,b\}}$ and $(uv)_{\{a,b\}} = u'_{\{a,b\}} * v'_{\{a,b\}}$, where the symbol $*$ denotes the juxtaposition of words. Note that $h = 1$ if and only if uv is a reduced factorization.

Lemma 2. *Let u and v be commuting elements of F_Γ such that uv is a reduced factorization, and let $\{a, b\}$ be a pair of non-adjacent vertices in Γ. Then there exists a word $h_{\{e\}}$ in the letters a, a^{-1}, b and b^{-1} such that*

$$\begin{aligned} u_{\{e\}} &= h_{\{e\}} * h_{\{e\}} * \cdots * h_{\{e\}}, \text{ with } j_e \text{factors} \\ v_{\{e\}} &= h_{\{e\}} * h_{\{e\}} * \cdots * h_{\{e\}}, \text{ with } k_e \text{factors}. \end{aligned}$$

Proof. Since uv is a reduced factorization, vu must also be one, since $|vu| = |uv| = |u| + |v|$. So we have that $u_{\{e\}} * v_{\{e\}} = (uv)_{\{e\}} = (vu)_{\{e\}} = v_{\{e\}} * u_{\{e\}}$ from which the result follows. □

Proposition 3. *Suppose that u and v are commuting elements in F_Γ such that uv is a reduced factorization, and the subgraph of Γ generated by* $\mathrm{supp}(uv)$ *has connected complement. Then there exists an element, h, in F_Γ such that both u and v belong to the cyclic subgroup of F_Γ generated by h.*

Proof. If $|u_{\{a,b\}}| = |v_{\{a,b\}}|$ for all pairs $e = \{a, b\}$ of non-adjacent vertices in Γ, then the Lemma 2 implies that $v_{\{e\}} = u_{\{e\}}$, for all e, hence $\mathrm{PB}(u) = \mathrm{PB}(v)$, $u = v$, and we are done.

On the other hand, if $j_{a,b} < k_{a,b}$, say, for some non-adjacent pair $\{a, b\}$, then the connectedness of the complement of Γ implies that $j_e < k_e$ for all non- adjacent pairs e in Γ. Thus every basic projection of u is a proper initial segment of the corresponding basic projection of v, and v has the reduced factorization $v = uv'$. Moreover, since both factors of the projection $v_{\{e\}}$ have the same support, u and v' satisfy all the conditions of the Lemma 2 and the result follows by induction on $|u| + |v|$. □

Let $u, v \in F_\Gamma$ such that there is no letter x such that we have simultaniously that $u = xu'x^{-1}$, $|u| = |u'| + 2$ and $v = xvx^{-1}$, $|v| = |v'| + 2$. Then we say that u and v are *pairwise cyclicly reduced.* It is easy to see that, given any two elements u and v, there is a unique element z such that $u = zu'z^{-1}$, $|u| = |u'| + 2|z|$, $v = xv'x^{-1}$, $|v| = |v'| + 2|Z|$, and such that u' and v' are pairwise cyclicly reduced.

LEMMA 3. *Let u and v be commuting elements in F_Γ such that $|uv| < |u|+|v|$, u and v are pairwise cyclicly reduced, and the graph generated by $L = \mathrm{supp}(u) \cup \mathrm{supp}(v)$ has connected complement. Let g and h be defined by the equations*

$$\begin{aligned} u = u'g = hu'' \quad &\text{with} \quad |u| = |u'| + |g| = |u''| + |h| \\ v = g^{-1}v' = v''h^{-1} \quad &\text{with} \quad |v| = |v'| + |g| = |v''| + |h| \\ uv = u'v' = v''u'' \quad &\text{with} \quad |uv| = |u'| + |v'| = |u''| + |v''|. \end{aligned}$$

Then $L = \mathrm{supp}(u) = \mathrm{supp}(v) = \mathrm{supp}(g) = \mathrm{supp}(h)$.

Using this lemma we may show

PROPOSITION 4. *Let u and v be commuting elements in F_Γ satisfying*

(1) $|uv| < |u| + |v|$
(2) *u and v are pairwise cyclicly reduced*
(3) *the induced subgraph of Γ on the set $L = \mathrm{supp}(u) = \mathrm{supp}(v)$ has connected complement.*

Then there exists an element h of F_Γ such that both u and v belong to the cyclic subgroup of F_Γ generated by h, $gp\langle h\rangle$.

Proof. We first show that vu^{-1} is a reduced factorization, then applying Proposition 3 will give the result. Suppose to the contrary that $|vu^{-1}| \leq |v| + |u^{-1}|$. Then there would be some letter, x, such that $u = rx$ and $v = sx$ are both reduced factorizations. By Lemma 3, $x \in \mathrm{supp}(g)$, so that $u = rx$ implies that both $g = g'x$ and $v = g^{-1}v' = x^{-1}s' = x^{-1}s''x$ are reduced factorizations. Similarly $x \in \mathrm{supp}(h)$, so $h = x^{-1}h'$ and $u = hu'' = x^{-1}r' = x^{-1}r''x$, also reduced, contradicting assumption [2]. □

Proof. (of Lemma 2)

(1) $\mathrm{supp}(g) = \mathrm{supp}(h)$; For any a in L we have $|(uv)_{\{a\}}| = |u_{\{a\}}| + |v_{\{a\}}| - 2|g_{\{a\}}|$ and $|(vu)_{\{a\}}| = |v_{\{a\}}| + |u_{\{a\}}| - 2|h_{\{a\}}|$, so $|g_{\{a\}}| = |h_{\{a\}}|$ for all a and $\mathrm{supp}(g) = \mathrm{supp}(h)$.
(2) $\mathrm{supp}(u') = \mathrm{supp}(u'')$; This follows since

$$|u_{\{a\}}| = |u'_{\{a\}}| + |g_{\{a\}}| = |u''_{\{a\}}| + |h_{\{a\}}|.$$

Similarly we have $\mathrm{supp}(v') = \mathrm{supp}(v'')$.
(3) $L = \mathrm{supp}(u) = \mathrm{supp}(v)$; Suppose to the contrary that, say, there is a vertex $b \in L - \mathrm{supp}(v)$. Since b is connected to $\mathrm{supp}(v)$ by a path of non-edges, we may assume that b is non-adjacent to a vertex $c \in \mathrm{supp}(v)$. Let $e = \{b, c\}$. We have that

$$\begin{aligned} u_{\{b,c\}} &= u'_{\{b,c\}} * g_{\{c\}} = h_{\{c\}} * u''_{\{b,c\}}, \\ v_{\{b,c\}} &= v_{\{c\}} = g^{-1}_{\{c\}} * v'_{\{c\}} = v''_{\{c\}} * h^{-1}_{\{c\}}, \text{ and} \\ (uv)_{\{b,c\}} &= (vu)_{\{b,c\}} = u'_{\{b,c\}} * v'_{\{c\}} = v''_{\{c\}} * u''_{\{b,c\}}, \end{aligned}$$

and we see that all the various factors in these three equations are non-trivial. The factors $u'_{\{b,c\}}$ and $u''_{\{b,c\}}$ are non-trivial since b is an element of $\mathrm{supp}(u)$. To see that $h_{\{c\}}$ and $v''_{\{c\}}$ are non-trivial let k denote the position of the first b in the projection $u_{\{b,c\}}$. Then by the third equation $k = |v''_{\{c\}}|+(k-|h_{\{c\}}|)$, and so $|v''_{\{c\}}| = |h_{\{c\}}|$. Since $v_{\{c\}}$ is non-trivial, both $v''_{\{c\}}$ and $h_{\{c\}}$ are non-trivial. It follows easily that the remaining factors are non-trivial.

Fact 1. *The equations*

$$\begin{aligned} u'_{\{x,y\}} * g_{\{x,y\}} &= h_{\{x,y\}} * u''_{\{x,y\}}, \\ g^{-1}_{\{x,y\}} * v'_{\{x,y\}} &= v''_{\{x,y\}} * h^{-1}_{\{x,y\}}, \text{ and} \\ u'_{\{x,y\}} * v'_{\{x,y\}} &= v''_{\{x,y\}} * u''_{\{x,y\}} \end{aligned}$$

with all factors non-trivial, imply that $u_{\{x,y\}} = z*u'''_{\{x,y\}}*z^{-1}$, *and* $v_{\{x,y\}} = z*v'''_{\{x,y\}}*z^{-1}$, *with* z *equal to one of* x, y, *or their inverses.*

Thus we have that $u_{\{b,c\}} = z*u'''_{\{b,c\}}*z^{-1}$ and $v_{\{b,c\}} = z*v'''_{\{c\}}*z^{-1}$, with z equal to either c or its inverse.

If c is a first letter of u, then we must have reduced factorizations $u = cxc^{-1}$ and $v = cyc^{-1}$, contradicting assumtion [2].

Suppose then that c is not the first letter of u, that is, there is a letter c_1 in $\mathrm{supp}(u)$ so that $\{c, c_1\}$ is not an edge in Γ and c_1 occurs first in $u_{\{c,c_1\}}$. Since c is both the first and the last vertex in $u_{\{b,c\}}$ and $v_{\{b,c\}}$, and all the factors $g_{\{b,c\}}$, $h_{\{b,c\}}$, $u'_{\{b,c\}}$, $v'_{\{b,c\}}$, $u''_{\{b,c\}}$, and $v''_{\{b,c\}}$ are non-trivial, it follows that the projections $g_{\{c\}}$, $h_{\{c\}}$, $u'_{\{c\}}$, $v'_{\{c\}}$, $u''_{\{c\}}$, and $v''_{\{c\}}$ are all non-trivial. Then we have that $g_{\{c,c_1\}}$, $h_{\{c,c_1\}}$, $u'_{\{c,c_1\}}$, $v'_{\{c,c_1\}}$, $u''_{\{c,c_1\}}$, and $v''_{\{c,c_1\}}$ are all non-trivial and we can apply the fact to get that $u_{\{c,c_1\}} = z*u'''_{\{c,c_1\}}*z^{-1}$, and $v_{\{c,c_1\}} = z*v'''_{\{c,c_1\}}*z^{-1}$, with z either c_1 or its inverse. We may now continue in this way to find c_2, c_3, etc. until we find a letter, c_k, which is a first letter of u. We must now have that $u = c_kxc_k^{-1}$ and $v = c_kyc_k^{-1}$ are reduced factorizations, contradicting assumption [2].

(4) $L = \mathrm{supp}(g)$; We define two subsets, C and S, of L. S, the set of *survivors*, is given by $S = L - \mathrm{supp}(g) = L - \mathrm{supp}(h)$ and C, the set of *casualties*, by $C = (L - \mathrm{supp}(u')) \cup (L - \mathrm{supp}(v'))$. No letter can be both a casualty and a survivor, since if, say, $x \in L - \mathrm{supp}(g)$ and $x \in L - \mathrm{supp}(u')$, then $x \in L - \mathrm{supp}(u)$, a contradiction. On the other hand every letter of L is either a casualty or a survivor, since to say that a is neither a casualty nor a survivor is to say that none of the projections, $g_{\{a\}}$, $h_{\{a\}}$, $u'_{\{a\}}$, $v'_{\{a\}}$, $u''_{\{a\}}$, or $v''_{\{a\}}$ is trivial, and we have seen that such an situation leads to a contradiction to the fact that u and v are pairwise cyclicly reduced. Thus the sets C and S partition the set L.

Claim 1. *Let* a *and* b *be non-commuting elements of* L. *Then either both* a *and* b *are elements of* S, *or both are elements of* C.

Proof. Let b be the last letter of $u_{\{a,b\}}$.

If b is a survivor, then b is not an element of $\mathrm{supp}(g)$ and so $u_{\{a,b\}} = u'_{\{a,b\}} * g\{a,b\} = u'_{\{a,b\}} * g_{\{a\}}$. Since b is the last letter we must have that $|g_{\{a\}}| = 0$, that is, a is a survivor also.

If b is a casualty, say b is not an element of $\text{supp}(u') = \text{supp}(u'')$, then $u_{\{a,b\}} = h_{\{a,b\}} * u''_{\{a\}}$, and since b is the last letter of $u_{\{a,b\}}$ we must have that $|u''_{\{a\}}| = 0$, that is, a is a casualty also. This proves the claim. □

Thus the complement of the induced subgraph $\Gamma(L)$ has been partitioned as a disjoint union of graphs

$$\Gamma(L)^c = \Gamma(C)^c \cup \Gamma(S)^c.$$

Since this graph is connected, one of the factors is empty, and since the product uv is not a reduced factorization, there are elements which do not survive. These elements must be casualties. Therefore there are no survivors, and $\text{supp}(g) = L$ as claimed.

This completes the proof of lemma 3. □

We may now give necessary and sufficient conditions for two elements, u and v, to commute in a graph group.

Let u and v be two pairwise cyclicly reduced elements which commute in F_Γ and let $L = \text{supp}(u) \cup \text{supp}(v)$. L induces a full subgraph $\Gamma(L)$ of Γ whose complement decomposes uniquely into connected components

$$\Gamma(L)^c = B_1 \cup B_2 \cup \cdots \cup B_k.$$

So every vertex in B_i is adjacent in Γ to every vertex in B_j for $i \neq j$. This induces unique reduced factorizations of u and v as $u = u_1 u_2 \cdots u_k$ and $v = v_1 v_2 \dots v_k$, where each of u_i and v_i is supported only by vertices in B_i and such that each pair u_i and v_i satisfies the conditions of either Proposition 3 or Proposition 4. Thus there exists elements h_i, $1 \leq i \leq j$, such that u_i and v_i belong to the cyclic subgroup of F_Γ generated by h_i.

We have proved the following

Proposition 5. *Let u and v be elements of F_Γ which are pairwise cyclicly reduced. Then u and v commute if and only if there are reduced factorizations $u = u_1 u_2 \dots u_k$, and $v = v_1 v_2 \dots v_k$, such that the following conditions hold:*

1. *there is an element h_i such that both u_i and v_i belong to the subgroup generated h_i.*
2. $\text{supp}(h_i)$ *is disjoint from* $\text{supp}(h_j)$ *for $i \neq j$,*
3. *every vertex in* $\text{supp}(h_i)$ *is adjacent to each vertex in* $\text{supp}(h_j)$*, $i \neq j$,*

Corollary 2. *If two elements, u and v, commute in a graph group, then any pair, h_i and h_j, defined above, also commutes.*

Corollary 3. *If u and v are pairwise cyclicly reduced and commute in F_Γ and if u is not cyclicly reduced, then one of the u_i's is not cyclicly reduced and thus $v_i = 1$.*

Corollary 4. *If two elements, u and v, commute, then we can find an element, p, such that pup^{-1} and pvp^{-1} are both cyclicly reduced.*

Let $u \in F_\Gamma$. and let $B_1, B_2, \dots, B_k$ denote the connected components of the complement of $\Gamma(\text{supp}(\text{CR}(u)))$ and write $\text{CR}(u) = u_1 u_2 \dots u_k$, such that $\text{supp}(u_i) = B_i$. Each u_i belongs to a maximal cyclic subgroup whose generator, h_i, is uniquely defined up to sign. We call h_i a *pure factor of u*, and denote the set of pure factors of u by $\text{PF}(u)$. Note that the pure factors of u are cyclicly reduced. We also define link(u) to be the set of all vertices in Γ which are not elements of supp(CR(u)) but which are adjacent to every vertex in supp(CR(u)).

Proposition 5 now implies

THEOREM 1 (The Centralizer Theorem). *The centralizer of an element* $u = p\mathrm{CR}(u)p^{-1}$ *is conjugate by* p *to the subgroup of* F_Γ *generated by* $\mathrm{link}(u) \cup \mathrm{PF}(u)$.

Moreover, the set $\mathrm{PF}(u) = \{h_1, \ldots, h_k\}$ *generates a free abelian group of rank* k *and*

$$\mathrm{cent}(\mathrm{CR}(u)) = \langle h_1\rangle \otimes \langle h_2\rangle \otimes \cdots \otimes \langle h_k\rangle \otimes F_{\Gamma(\mathrm{link}(u))}.$$

Recall that a graphical subgroup of a graph group F_Γ is a subgroup which is generated by a (full) subgraph of Γ. It is a consequence of the centralizer theorem that the centralizer of any element of a graph group is itself a graph group and is, in fact, isomorphic to a graphical subgroup. It is not in general true that every subgroup of a graph group is itself a graph group.

5. Automorphisms of Graph Groups

5.1. **Elementary Automorphisms.** One striking similarity between free and free abelian groups is that their automorphism groups have the same generators. If F is free of rank n, then aut(F) is generated by the Neilsen transformations, which, written additively, generate $\mathrm{aut}(F/[F,F]) = \mathrm{GL}(n,Z)$ in the form of the elementary matrices.

Let F be a free group and let S be a basis for F. The Nielsen transformations are as follows:

- **[N1]:** *Inversion:* ι_x for some $x \in X$ with $\iota_x(x) = x^{-1}$ and $\iota_x(z) = z$ for $z \in (X - x)$,
- **[N2]:** *Transposition:* $\sigma_{x,y}$ for $x, y \in X$, $\sigma_{x,y}(x) = y$, $\sigma_{x,y}(y) = x$ and $\sigma_{x,y}(z) = z$ for $z \in (X - \{x,y\})$,
- **[N3]:** *Transvection:* $\tau_{x,y}$ for $x, y \in X$, $\tau_{x,y}(x) = xy$, $\tau_{x,y}(z) = z$ for $z \in (X - x)$.

Strictly speaking, the transpositions are redundant.

Since a graph group is a generalization of both the free and free abelian groups, it is natural to expect that some variation on this theme will hold true.

Replacing S with the set V of vertices of Γ, the inversions also define automorphisms of the graph group F_Γ. Many of the transpositions and transvections, however, fail to define endomorphisms of F_Γ.

Given a transvection $\tau : x \to xy$, τ defines an automorphism of F_Γ if and only if y is adjacent to every vertex to which x is adjacent. Following [5], we define a partial order on the vertex set of a graph by setting $y \sqsupset x$ if given any other vertex, z, distinct from x and y and which is adjacent to x, then z is also adjacent to y, and we say that b *dominates* a. Thus the transvection τ defines an automorphism of F_Γ if and only if y dominates x, and we call τ a *dominated transvection.*

A Transposition $x \leftrightarrow y$ defines an endomorphism, hence an automorphism, if and only if $x \sqsupset y$ and $y \sqsupset x$, however such transpositions are not sufficient in general to generate those automorphisms of F_Γ which simply permute the vertices. Fortunately these automorphisms are neatly described as those automorphisms of F_Γ which are induced by automorphisms of the graph Γ, which will be call *graphic* automorphisms.

The inversions, dominated transvections, and graphic automorphisms do not in general generate $\mathrm{aut}(F_\Gamma)$. For example, if Γ is the pentagon of Figure 1 then

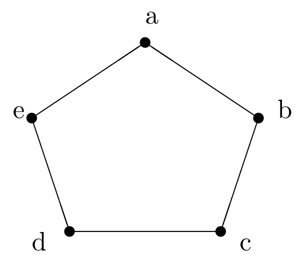

FIGURE 1.

the order relation $\sqsupset$ on the vertices is trivial, so that there are no dominated transvections at all. The inversions and graphic automorphisms generate a finite group of automorphisms all of which preserve the length of the elements of F_Γ, in particular, containing no inner automorphisms of F_Γ.

Augmenting this list of automorphisms to include those inner automorphisms corresponding to conjugation by vertices of Γ will not serve in general to generate $\mathrm{aut}(F_\Gamma)$. For example, consider the graph consisting of two pentagons joind at a vertex. Again, the order relation $\sqsupset$ is trivial for this graph so that there are no dominated transvections. Define an automorphism as in Figure 2. It is clear that

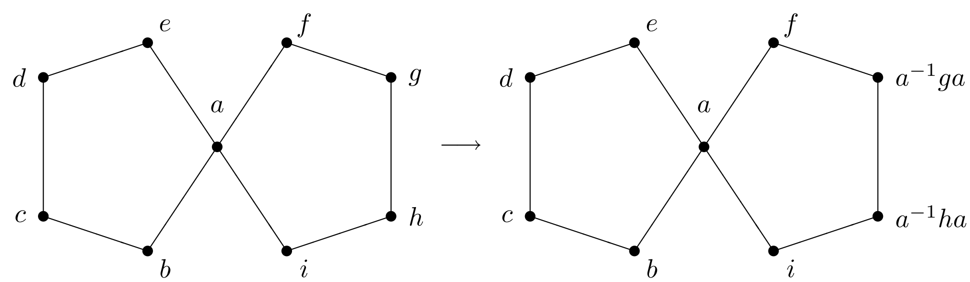

FIGURE 2. A Locally Inner Automorphism

this cannot be obtained from inner, inverting and graphic automorphisms. We will call such an automorphism locally inner.

Recall that for any vertex v in Γ, $\mathrm{link}(v) = \{x \in V \mid (v, x) \in E\}$, and $\mathrm{star}(v) = \mathrm{link}(v) \cup \{v\}$. Let C be any connected component of $\Gamma - \mathrm{star}(v)$. We define a map $\lambda = \lambda_{v,Y}$ from V into F_Γ by setting

$$\begin{aligned} \lambda(x) &= vxv^{-1} \text{ if } x \text{ is a vertex in } C, \text{ and} \\ \lambda(x) &= x \text{ if } x \text{ is not a vertex in } C. \end{aligned}$$

Note that λ defines an automorphism of F_Γ and that there are only finitely many such λ. We call λ a *locally inner automorphism.*

DEFINITION 1. *We define the set of* elementary automorphisms *of F_Γ to consist of the following:*

- **[EN1]:** Inversion: ι_x *for some* $x \in V$ *with* $\iota_x(x) = x^{-1}$ *and* $\iota_x(v) = v$ *for* $v \in (V - x)$,
- **[EN2]:** Graphic Automorphism: g_ϕ *for* $\phi \in \mathrm{aut}(\Gamma)$, $g_\phi(v) = \phi(v)$ *for all* $v \in V$,
- **[EN3]:** Dominated Transvection: $\tau_{x,y}$, *for* $x, y \in V$ *such that* $(x, z) \in E \Rightarrow (y, z) \in E$, $\tau_{x,y}(x) = xy$, $\tau_{x,y}(v) = v$ *for* $v \in (V - x)$,
- **[EN4]:** Locally Inner Automorphism: $\lambda_{x,C}$, *where* $x \in V$ *and* C *is a connected component of* $\Gamma - \mathrm{star}(x)$, *with* $L(v) = x^{-1}vx$ *for* $v \in C$, $L(v) = v$ *for* $v \in V - C$.

Note that the condition in [EN4] we may require that $|C| > 1$, since if C consists of a single vertex then that vertex is dominated by v, so that $\lambda_{v,C}$ can be written in terms of dominated transvections and inversions.

Conjecture 1. *The elementary automorphisms generate* $\mathrm{aut}(F_\Gamma)$.

We will prove some results in support of Conjecture 1. In particular, we show that it is true if Γ is a tree or, on the other hand, if Γ is star two-connected and has girth greater than four. We will also show that the collection of graph groups for which the Conjecture 1 is true is closed under the operations of free product and direct sum.

Note that, in contrast to the Nielsen automorphisms for a free group, the graphic automorphisms are not redundant, as can be seen in the case of the pentagon.

5.2. **EN-Equivalence.**

Definition 2. *Let* f' *and* f'' *be two automorphisms of* F_Γ. *We say that* f' *and* f'' *are* EN-*equivalent if* $f'h' = h''f''$, *where* h' *and* h'' *are both products of elementary automorphisms.*

This is an equivalence relation on $\mathrm{aut}(F_\Gamma)$ and we would like to show that every automorphism of F_Γ is EN-equivalent to the identity.

As in the case of free groups, it is convenient to work with bases rather than the automorphisms themselves, and while for a free group a basis is just a generating set of minimal cardinality, a basis in a graph group is in fact a graph. In precise analogy with free and free abelian groups, we define a functor U from the category of groups to the category of graphs which sends each group G to the graph $U(G)$, called the *underlying graph of* G, whose vertex set is G itself and in which two elements are adjacent if and only if they commute. U has a left adjoint F which sends each graph Γ to F_Γ, the graph group on Γ. The adjoint isomorphism is

$$\phi : \mathrm{hom}(F_\Gamma, G) \to \mathrm{hom}(\Gamma, U(G)), \tag{1}$$

and the elements of $\mathrm{hom}(\Gamma, U(F_\Gamma))$ corresponding $\phi(\mathrm{aut}(F_\Gamma))$ are called *bases* of the graph group F_Γ, with the basis corresponding to the identity called the *preferred basis* of F_Γ.

It is clear by examining abelianizations that any two bases for a graph group must have the same number of vertices, which number has been defined to be the *rank of the graph group.* It has been shown, see [3], that any two bases of a graph group are isomorphic.

Definition 3. *Two bases of* F_Γ *are* EN-*equivalent if one can be transformed into the other by a finite sequence of moves of the following forms, where* $\phi : \Gamma \to U(F_\Gamma)$ *is a basis of* F_Γ

[EN1]: *replace $\phi(v)$ with $\phi(v)^{-1}$ for some v in Γ,*
[EN2]: *replace each $\phi(v)$ with $\phi(\sigma(v))$ for some σ in $\mathrm{aut}(\Gamma)$,*
[EN3]: *replace $\phi(v)$ with $\phi(v)\phi(w)$ for some pair of verices v and w in Γ such that w dominates v.*
[EN4]: *replace $\phi(v)$ with $\phi(a)\phi(v)\phi(a)^{-1}$ for each vertex v in some connected component C of $\Gamma - \mathrm{star}(a)$ for some fixed vertex a in Γ,*
[EN5]: *replace each $\phi(v)$ with $h(\phi(v))$ for some elementary automorphism h of F_Γ.*

FACT 2. *Two automorphisms are* EN-*equivalent if and only if their corresponding bases are* EN-*equivalent.*

FACT 3. *A basis is* EN-*equivalent to Γ if and only if its corresponding automorphism can be written as the product of elementary automorphisms.*

5.3. **General Reduction Theorems.**

PROPOSITION 6. *If two bases are conjugate, then they are* EN-*equivalent.*

Proof. Let v be a vertex in Γ and let $X_1, \ldots, X_k$ denote the vertices of the connected components of $\Gamma - \mathrm{star}(v)$. Then the product

$$\lambda_{v,X_1}\lambda_{v,X_2}\ldots\lambda_{v,X_k}$$

of elementary automorphisms transforms the basis V into the basis $v^{-1}Vv$, so that if two bases are conjugate by a vertex then they are EN-equivalent, and the result follows. □

The partial order $\sqsupset$ that has been defined induces an equivalence relation on the vertex set, and we denote the equivalence class of a by $[a]$, $[a] = \{v \in V : a \sqsupset v \text{ and } v \sqsupset a\}$. Each equivalence class must generate a subgraph which is either complete or discrete and we will abuse terminology and say that an equivalence class is complete or discrete.

Let D denote the set of vertices in Γ which are dominated by a non-adjacent vertex and let C denote the set of vertices in Γ which are adjacent to every vertex dominating them, so $C = V - D$.

FACT 4. *D contains every vertex which belongs to a non-singleton discrete equvalence class in Γ. C contains every vertex which belongs to a non-singleton complete equivalence class in Γ.*

Proof. The first statement is immediate. For the second, suppose that v is both equvalent and adjacent to v'. Let w be any vertex dominating v. Then w dominates v' also, and so is adjacent to v since v' is. Thus v belongs to C. □

FACT 5. *No element of D domminates any element of C.*

Proof. Suppose to the contrary that we have. $v \in C$, $w \in D$, and $w \sqsupset v$. w is dominated by some vertex, x, which is not adjacent to w. Since both x and w dominate v, both are adjacent to v. Since w dominates v, w is adjacent to x, a contradiction. □

FACT 6. *If v is an element of C, then the set of vertices dominating v,* $\mathrm{Dom}(v)$, *generates a complete subgraph of Γ.*

PROPOSITION 7. *Every basis of* F_Γ *is* EN*-equivalent to a basis,* θ*,* $\theta : \Gamma \to U(F_\Gamma)$*, such that every vertex in* $\text{supp}(\text{CR}(\theta(v)))$ *dominates* v*, with at least one vertex equivalent to* v*. Moreover, either*

1. $v \in C$
2. $\text{cent}(\text{CR}(\theta(v))) = \text{cent}(v) = F_{\Gamma(\text{star}(v))}$
3. *Every pure factor of* $\theta(v)$ *is supported by a single vertex,*

or

1. $v \in D$
2. *Every pure factor of* $\theta(v)$*, except perhaps* h_1*, is supported by exactly one vertex which strictly dominates* v *and is adjacent to* v*,*
3. *No vertex in* $\text{supp}(x')$ *is adjacent to* v*.*
4. $\text{cent}(\text{CR}(\theta(v))) = \text{cent}(h_1 = \langle h_1 \rangle \otimes F_{\Gamma(\text{link}(h_1))}$

Proof. In what follows let f be an automorphism of F_Γ, let p be a vertex of Γ and let $\pi = \text{CR}(f(p))$.

LEMMA 4. *If* $x \in \text{supp}(\pi)$ *then* $\text{val}(x) \geq \text{val}(p)$*.*

Proof. We have seen that the centralizer of $f(p)$ is conjugate to the subgroup $\text{cent}(\pi) = \langle h_1 \rangle \otimes \langle h_2 \rangle \otimes \cdots \otimes \langle h_k \rangle \otimes F_{\Gamma(\text{link}(\pi))}$, where $\text{PF}(\pi) = \{h_1, \cdots, h_k\}$. So $\text{cent}(\pi)$ is a graph group of rank $k + |\text{link}(\pi)|$. But $\text{rank}(\text{cent}(p)) = \text{val}(p) + 1$, so $\text{val}(p) + 1 = k + |\text{link}(\pi)|$.

Let $x \in \text{supp}(\pi)$, then $x \in \text{supp}(h_i)$ for some i, and x must be adjacent to at least one vertex corresponding to each of the $k - 1$ other pure factors, and also to each of the vertices in $\text{link}(\pi)$. Thus $\text{val}(x) \geq k - 1 + |\text{link}(\pi)| = \text{val}(p)$ and the lemma is proved. □

LEMMA 5. *There exists a vertex,* x*, in* $\text{supp}(\pi)$ *such that* $\text{val}(x) = \text{val}(p)$*.*

Proof. We examine the induced map, f', on the abelianization of F_Γ. Order V such that the valences of succesive vertices is non-increasing, and represent f' by the square matrix M. Let r denote the number of vertices of valence greater than $\text{val}(p)$. The first r columns of M consist only of zeros beyond the r'th row. If every vertex in $\text{supp}(\pi)$ had valence greater than $\text{val}(p)$ then the column corresponding to p would also have zeros below the r'th row, giving $r + 1$ such rows, a contradiction. □

LEMMA 6. *The vertices of valence* $\text{val}(p)$ *in* $\text{supp}(\pi)$ *belong to a single equivalence class. Moreover, let* α *be a vertex of valence* $\text{val}(p)$ *in* $\text{supp}(\pi)$*. The pure factors for* $f(p)$*,* $\text{PF}(f(p)) = \{h_1, \ldots, h_k\}$*, satisfy one of the following sets of conditions: either;*

1. $|\text{supp}(h_1)| > 1$*,*
2. *If* $\alpha \in \text{supp}(h_1)$*,* $\text{val}(\alpha) = \text{val}(p)$*, then every vertex in* $\text{supp}(h_1)$ *dominates* α *and is non-adjacent to* α*,*
3. $h_i, i = 2, \ldots, k$*, is supported by one vertex and that vertex strictly dominates* α *and is adjacent to* α*, and*
4. $\text{cent}(\pi) = \text{cent}(h_1)$*,*

or;

1. $|\text{supp}(h_1)| = 1$ *for all* $i = 1, \ldots, k$*,*
2. *for all* $\beta \in \text{supp}(\pi)$*,* π *dominates* α *and is adjacent to* α*, and*
3. $\text{cent}(\pi) = \text{cent}(\alpha)$*.*

Proof. As before, let the pure factors of π be denoted by $h_1, h_2, \ldots, h_k$.

Case 1: h_1 is supported by more than one vertex. In this case every vertex in $\text{supp}(\pi)$ whose valence is the same as that of p must lie in $\text{supp}(h_1)$ as well. Since if α is a vertex in h_i, $i > 1$, then α is adjacent to at least two vertices in $\text{supp}(h_1)$, at least $k-2$ vertices corresponding to the other pure factors, and every vertex in $\text{link}(\pi)$. Altogether,

$$\text{val}(\alpha) \geq 2 + k - 2 + |\text{link}(\pi)| = \text{val}(p) + 1.$$

So every vertex in $\text{supp}(\pi)$ with valence $\text{val}(p)$ is contained in $\text{supp}(h_1)$. But every vertex in $\text{supp}(h_1)$ is adjacent to at least

$$|\text{supp}(h_2)| + \cdots + |\text{supp}(h_k)| + |\text{link}(\pi)|$$

vertices, that is,

$$|\text{supp}(h_2)| + \cdots + |\text{supp}(h_k)| + \text{val}(p) + 1 - k$$

vertices.

Since there is, by Lemma 5, a vertex in h_1 with valence $\text{val}(p)$, it follows that $|\text{supp}(h_2)| = \cdots = |\text{supp}(h_k)| = 1$, and the pure factors $h_2, \ldots, h_k$ consist of individual vertices. Moreover, we have identified all $\text{val}(p)$ vertices which are adjacent to a vertex of valence $\text{val}(p)$ in $\text{supp}(h_1)$. They consist of $\text{link}(\pi)$ together with $h_2, \ldots, h_k$. So if α is a vertex of valence $\text{val}(p)$ in $\text{supp}(h_1)$ then every other vertex in $\text{supp}(h_1)$ dominates α and no vertex in $\text{supp}(h_1)$ is adjacent to α. In particular all vertices in $\text{supp}(\pi)$ with valence $\text{val}(p)$ are equivalent, and $[\alpha]$ is discrete if $|\text{supp}(h_1| > 1$.

We see also that the vertices supporting $h_2, \ldots h_k$ all strictly dominate α and are adjacent to α.

Case 2: Each pure factor h_1, $2 \leq i \leq k$, is supported by a single vertex.

Then h_i is adjacent to h_j for all $j \neq i$, and also with each vertex of $\text{link}(\pi)$, a total of $(k-1) + |\text{link}(\pi)| = \text{val}(p)$ vertices. So if h_i is a vertex of valence $\text{val}(p)$, then every h_j dominates h_i for all j. In particular, those h_j's of valence $\text{val}(p)$ are all equivalent to h_i.

Note that, in this case, the equivalence class of α is complete if there is more than one h_i with valence $\text{val}(p)$.

□

Thus we can associate to each vertex p in Γ the equivalence class of those vertices of Γ in $\text{supp}(\text{CR}(f(p)))$ which have valence $\text{val}(p)$. This defines a map, $\mathcal{F}$, from the vertex set of Γ to the vertex set of the quotient graph $\Gamma/[-]$. Following Kim [5], or Droms [3], it can be shown that $\mathcal{F}$ is a graph morphism and lifts to a graph automorphism of Γ. Applying the graphic move corresponding to the inverse of this automorphism gives a basis with the desired properties. This completes the proof of the Proposition 7. □

Proposition 8. *Every basis of F_Γ is* EN*-equivalent to a basis, θ, which satisfies the conclusion of Proposition 7, and is such that for each vertex v in C we have that $\theta(v)$ is conjugate to v. Moreover, the set $[v]$ is congugate to the set $[\theta(v)]$.*

Proof. Let θ be a basis satisfying the conclusion of proposition 8, and let v be a vertex in C. Suppose that the proposition is true for all vertices in C of valence greater than that of v.

We assume that $\theta(v)$ is cyclicly reduced. Then if w is adjacent to v, we have that every vertex in $\mathrm{supp}(\theta(w))$ is in $\mathrm{star}(v)$. If w dominates v, then w is in C and every vertex in $\mathrm{supp}(\mathrm{CR}(\theta(w)))$ dominates v, since w dominates v and every vertex in $\mathrm{supp}(\mathrm{CR}(\theta(w)))$ dominates w. Since each vertex dominating v is adjacent to v, it follows that $\theta(w)$ is cyclicly reduced. So if w srictly dominates v, then $\phi(w) = w$.

Consider the set, $\mathrm{Dom}(v)$, of vertices which dominate v. $\mathrm{Dom}(v)$ contains $[v]$ together with those vertices which strictly dominate v. $\mathrm{Dom}(v)$ generates a complete graph in Γ. We may now apply dominated transvections to assume that $\mathrm{supp}(\theta(v'))$ is contained in $[v]$ for each v' equivalent to v. There is now a reduction of $\theta(v)$ until $\theta(v) = v$ for all vertices v in $[v]$ by the result for free abelian groups. □

Proposition 9. *Every basis of F_Γ is* EN-*equivalent to a basis, θ, which satisfies the conclusion of Proposition 8 and is such that $\theta(v)$ contains only one pure factor for each vertex in Γ.*

Proof. First modify the basis until it conforms with the conclusions of the previous proposition. Then those $\theta(v)$ having more than one pure factor correspond to vertices in Γ outside C. The reduction procedes in order of decreasing valence. If the valence of v is maximal among all vertices in Γ, then there is nothing to show.

Assume that $\theta(v)$ is cyclicly reduced with more than one pure factor and that $\theta(w)$ contains only one pure factor for each vertex of Γ whose valence is greater than that of v. Except for the first pure factor, which is supported by some vertices equivalent to v itself, the pure factors of $\theta(v)$ are supported by indiviual vertices which dominate v and are adjacent to v. Such vertices must belong to C, and so if w is such a vertex, we have that $\theta(w)$ is conjugate to w. But since w is adjacent to v, it must be a conjugate by a word whose support is contained in $\mathrm{link}(v)$, so that in fact $\theta(w) = w$, and a sequence of dominated transvections will remove w from among the pure factors of $\theta(v)$.

In this way we can remove the remaining superfluous pure factors in the basis θ. □

In sum we have

Theorem 2. *Every basis of F_Γ is* EN-*equivalent to a basis, θ, such that for each vertex, v, in Γ, we have*

1. *if v is an element of C, then $\theta(v)$ is conjugate to v itself, and moreover $[v]$ is conjugate to the set $[\theta(v)]$.*
2. *if v is an element of D, then $\theta(v)$ has exactly one pure factor which is supported entirely by vertices which dominate v and are not adjacent to v, some of which must be equivalent to v.*

Let Σ denote the collection of graphs such that their graph groups satisfy the Conjecture 1. We know from the classical theory that Σ contains the complete and discrete graphs.

Given two graphs Γ and Ω, their *disjoint union* is the graph $\Gamma \cup \Omega$ which has vertex set the disjoint union of the vertex sets of Γ and Ω and whose edge set is the disjoint union of their edge sets. The *join* of Γ and Ω is the graph $\Gamma + \Omega = (\Gamma^c \cup \Omega^c)^c$, that is, the join is obtained from the disjoint union by connecting every vertex in the first factor with every vertex in the second factor.

Theorem 3. *Σ is closed under $\cup$ and $+$.*

That is to say, if two groups satisfy the Conjecture 1, then their free product and their direct sum do as well.

Proof. For the join, let Γ have join components $\{\Gamma_i : i = 1, \ldots, k\}$, with each Γ_i an element of Σ. Since the singleton components play a special role, let us choose notation so that they all occur first in the list $\{\Gamma_i\}$, so there is a $j \leq k$ so that Γ_i is a singleton for all $i \leq j$. These singleton components will all be equivalent in Γ and will also be of maximal valence in Γ. Thus, if θ is a basis of F_Γ, we have that θ is EN-equivalent to a basis, which we will also call θ, with $\theta(v) = v$ for each vertex v making up an entire join component of Γ. Furthermore, since such a v dominates all the other vertices in Γ, we may apply dominated transvections to assume that v is not an element of $\mathrm{supp}(\theta(w))$ for $w \neq v$.

Now, let x be a vertex in a non-singleton join component of Γ. The pure factors of $\theta(x)$ are supported by vertices dominating x, since the only vertices which both dominate x and belong to a different join component are precisely those vertices making up singleton join components. But we have already applied dominated transvections to assume that $\mathrm{CR}(\theta(x))$ is supported entirely by the component containing x. Since each Γ_i is in Σ, the result follows.

In the case of the union, let Γ have the connected components $\{\Gamma_i : i = 1, \ldots k\}$, with Γ_i a singleton component if and only if $i \leq j$ for some $j \leq k$, as before.

Let θ be a basis of F_Γ. Modify θ by EN moves until it satisfies the conclusions of Theorem 2. If v and w are vertices in different connected components and v dominates w, then w must be isolated in Γ. Thus, if x is not isolated in Γ, we must have that the pure factors of $\theta(x)$ are supported entirely by the connected component of x. We may apply locally inner moves to assume that $\theta(x)$ is cyclicy reduced. It follows that for every vertex, y, in the component of x, we have that $\theta(y)$ is supported entirely by that component. Since the component is in Σ, we may assume that $\theta(y) = y$.

We procede thusly for each non-singleton component of Γ until we have modified θ until it is the inclusion on each vertex of Γ except perhaps those vertices which are isolated. The isolated vertices, however, are all equivalent and are dominated by every vertex in Γ, and the result follows from Nielsen. □

Theorem 4. *Let ϕ be a basis for F_Γ satisfying theorem 2. Let v be a vertex and let Y be a connected component of $\Gamma - \mathrm{star}(v)$. Then if $v \in \mathrm{supp}(\phi(y))$ for some $y \in Y$, then $v \in \mathrm{supp}(\phi(x))$ for all $x \in Y$.*

Proof. We show that $v \in \mathrm{supp}(\phi(x))$ for some x adjacent to y and not adjacent to v, and the result will follow by connectedness of Y. $\phi(y)$ commutes with $\phi(x)$, so that there is a unique element z such that $\phi(y)$ and $\phi(x)$ have the reduced factorizations $\phi(y) = zy'z^{-1}$ and $\phi(x) = zx'z^{-1}$ with x' and y' pairwise cyclically reduced. The vertex v does not support either x' or y', since that would imply that v is adjacent y or x respectively. So v supports z, hence v supports $\phi(x)$. □

5.4. **A Theorem of Free Groups.** To work some examples, we need to examine closely the case of bases of a free group. Theorem 5, which follows, generalizes a theorem in [4].

Definition 4. *A set B in a free group with preferred basis, X, is said to be* Nielsen reduced *if for any three elements b_1, b_2, and b_3 in $B \cup B^{-1}$ we have that*

(n1): $b_i \neq 1$

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(n2): $b_ib_j \neq 1 \Longrightarrow |b_ib_j| \geq |b_i|, |b_j|$
(n3): $b_1b_2 \neq 1, b_2b_3 \neq 1 \Longrightarrow |b_1b_2b_3| > |b_1| - |b_2| + |b_3|$.

THEOREM 5. *4 Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots q_m\}$ be finite sets and set $D = P \cup Q$.*

Let $F(D)$ denote the free group on the set D, and let $\theta(D) = \{\theta(d) : d \in D\}$ be a new basis for $F(D)$ such that $\theta(q)$ is conjugate to q for each q in Q.

Then $\theta(D)$ can be Nielen reduced by a finite sequence of the following moves and their inverses

[M1]: *Replace $\theta(p)$ with $\theta(p)\theta(d)$, or $\theta(d)\theta(p)$, for some $p \in P$, and some d, $d \neq p$, $d \in D$.*
[M2]: *Replace $\theta(q)$ with $\theta(d)\theta(q)\theta(d)^{-1}$, for some $q \in Q$, and $d \in D$.*

Furthermore, if for some $d \in D$, we have $\theta(d) = d$, then $\theta(d)$ is not disturbed in the reduction.

Proof. Choose a well-ordering, $>$, of the set $D \cup D^{-1}$. This induces a well-ordering, which is also denoted by $>$, on the set $F(D)$, whose elements are identified with the reduced words which represent them, as follows: Set $w > w'$ if either $|w| > |w'|$, or $|w| = |w'|$ and and w is greater than w' in the lexicographical ordering of the set of reduced words of length $|w|$.

If w is a word, we define the *left half of* w, $L(w)$, to be the initial segment of w of length $\frac{1}{2}(|w| + 1)$, where $[t]$ denotes the integer part of t.

Now, define a new well ordering, denoted by $\gg$, on the set $F(D)$ by setting $y \ll z$ if either;

$$\min\{L(y), L(y^{-1})\} < \min\{L(z), L(z^{-1})\},$$

or

$$\min\{L(y), L(y^{-1})\} = \min\{L(z), L(z^{-1})\}, \quad \text{and}$$
$$\max\{L(y), L(y^{-1})\} < \max\{L(z), L(z^{-1})\}.$$

Note that if $|y| < |z|$ then $y \ll z$.

We shall show that if $\theta(D)$ is not Nielsen reduced, then we can apply one of the moves [M1] or [M2] to strictly reduce the set $\theta(D)$ in the well ordering $\gg$.

There are a number of cases. Let p, q and d be elements of $P \cup P^{-1}$, $Q \cup Q^{-1}$, and $D \cup D^{-1}$ respectively.

(1) If $|\theta(p)\theta(d)| < |\theta(p)|$ or $|\theta(d)\theta(p)| < |\theta(p)|$, then we may use one of M[1] or M[2] to replace $\theta(p)$ with $\theta(p)\theta(d)$ or $\theta(p)\theta(q)$ respectively, strictly reducing $\theta(D)$ with respect to $>$, hence with respect to $\gg$.
(2) If (n1) is satisfied for all appropriate elements of D and $|\theta(p)\theta(q)| < |\theta(q)|$ or $|\theta(q)\theta(p)| < |\theta(q)|$, then, since less than half of $\theta(q)$ cancels in the product, and since $\theta(q) = \alpha q \alpha^{-1}$ for some α, we have that $|\theta(p)\theta(q)\theta(p)^{-1}| < \theta(q)$ or $|\theta(p)^{-1}\theta(q)\theta(p)| < |\theta(q)|$ respectively, and a move of type [M2] will reduce $\theta(D)$ with respect to $\gg$.
(3) If $|\theta(q)\theta(q')| < |\theta(q)|$, then there are two cases;
   (a) $|\theta(q)\theta(q')| \geq |\theta(q')|$, so that less than half of $\theta(q)$ cancels in the product. Since $\theta(q) = \alpha q \alpha^{-1}$ for some α, we must have that $|\theta(q')\theta(q)\theta(q')^{-1}| < |\theta(q)|$ so that a move of type [M2] will reduce $\theta(D)$.
   (b) $|\theta(q)\theta(q')|$is less than both $|\theta(q)| and |\theta(q')|$. We may write $\theta(q) = \alpha q \alpha^{-1}$ and $\theta(q') = \beta q' \beta^{-1}$. Since more than half of both $\theta(q)$ and $\theta(q')$ cancel in their product, we must have $|\alpha| \neq |\beta|$, since, otherwise,

$\alpha = \beta$ and thus $q^{-1} = q'$, a contradiction. Suppose w.l.o.g. that $|\beta| > |\alpha|$ and set $\beta = \alpha q \delta$, reduced as written, for some δ. Then $(\alpha q^{-1} \alpha^{-1})(\alpha q \delta q' \delta^{-1} q^{-1} \alpha^{-1})(\alpha q \alpha^{-1}) = \alpha \delta q' \delta^{-1} \alpha^{-1}$ (not necessarily reduced as written), so that $|\theta(q)\theta(q')\theta(q)^{-1}| < |\theta(q')|$ and a move of type [M2] reduces $\theta(D)$ with respect to $\gg$.

(4) $|\theta(d)\theta(d')| \geq |\theta(d)|, |\theta(d')|$ for all elements d and d' in $D \cup D^{-1}$. Then we consider a triple $x, y, z \in D \cup D^{-1}$, with $xy \neq 1$, and $yz \neq 1$. Since no more than half of y cancels in either of the two products xy and yz, we may write $x = x'\alpha^{-1}$, $y = \alpha y' \beta^{-1}$, and $z = \beta z'$ such that the products $x'y'$ and $y'z'$ are both reduced as written. There are two cases.
   (a) If $b \neq 1$, then $xyz = x'y'z'$ is reduced as written, so that $|xyz| = |x| - |y| + |z| + 2|y'| > |x| - |y| + |z|$ and x, y, and z satisfy Nielsen's conditions.
   (b) If $b = 1$, then since less than half of y cancels in either of the two products, we must have that $|\alpha| = |\beta| = \frac{1}{2}|y| < \frac{1}{2}|x|, \frac{1}{2}|z|$, and that $p \neq q$.
   We have $\beta \neq \alpha$, so let us suppose that $\alpha \ll \beta$. There are two cases.
      (i) $z \in P \cup P^{-1}$, in which case $yz = \alpha z' \ll \beta z$, so that a move of type [M1] will reduce $\theta(D)$.
      (ii) $z \in Q \cup Q^{-1}$, in which case, since $|\beta| \neq \frac{1}{2}|z|$, we may write $z = \beta z'' \beta^{-1}$, reduced as written, for some z''. Now $yxy^{-1} = \alpha z'' \alpha^{-1} \ll \beta z'' \beta^{-1} = z$, so that a move of type [M2] reduces $\theta(D)$.

Note that for any element d in D, none of the moves described above increase the length of $\theta(d)$, so that, in particular, if $\theta(d) = d$, then it is left undisturbed in the reduction process.

This completes the proof of Theorem 5. □

5.5. **Automorphisms of a Tree Group.** In this section we will show that the automorphism conjecture is true if the graph is a tree. We remark that it has been shown by Droms [2] that a graph group is the fundamental group of a three manifold if and only if every component of the graph is a tree or a triangle, so this result will establish the Conjecture 1 for graph groups which are also three manifold groups.

Let $T = (V, E)$ be a tree, and partition V by $V = P \cup Q$, where P is the set of pendent vertices in Γ, i.e. those vertices of valence one.

The partial order we have defined on the vertices of a graph is particularly simple in the case of a tree. If v and w are vertices in T and v dominates w, then w must be pendent. If p is a pendent vertex and p is adjacent to v, then the set of vertices dominating p consists of every vertex in $\mathrm{star}(v)$.

Theorem 6. *The automorphism group of F_T is generated by elementary automorphisms.*

Proof. Let θ be a basis for F_T.

Since T is a tree, the only vertices not belonging to the set C, defined in section 2, are the pendent vertices. Thus by Theorem 2 we have that θ is EN-equivalent to a new basis, θ, such that $\theta(v)$ is conjugate to v for every non-pendent vertex, v, and for every pendent vertex, p, with p adjacent to v, we have that $\mathrm{supp}(\mathrm{CR}(f(p)))$ is contained in $\mathrm{link}(v)$.

Let v be a non-pendent vertex. By applying an inner automorphism, we may assume that $\theta(v) = v$. Then we will have that $\mathrm{supp}(\theta(y))$ is contained in $\mathrm{link}(v)$ for each vertex y in $\mathrm{link}(v)$. Notice that, since T has no circuits, $\mathrm{link}(v)$ generates a free subgroup of F_T, hence so does $\theta(\mathrm{link}(v))$, and the theorem will follow from theorem 5. The moves [M1] and [M2] of the theorem are restrictions of elementary moves to $\theta(\mathrm{link}(v))$. $\theta(\mathrm{link}(v))$ is a basis for the free group whose preferred basis is $\mathrm{link}(v)$. Since, see Lyndon [7], chapt 1, the length of an element of a Nielsen reduced basis is exactly 1, applying the theorem reduces the length of every basis element in $\theta(\mathrm{star}(v))$ to 1. If w is a non-pendent vertex in $\mathrm{star}(v)$, then $\theta(w) = w$. It is clear, then, that there is a sequence of graphic and inverting moves such that $\theta(p) = p$ for any pendent vertex in $\mathrm{star}(v)$. Thus θ can be reduced to the identity on $\mathrm{star}(v)$.

The non-pendent vertices of a tree form a connected set, (also a tree), so we can continue this process to reduce θ to the identity on all of T.

This completes the proof of theorem 6. □

5.6. **Automorphisms of a Star Two-Connected Graph Group.** If a graph Γ such that the deletion of any vertex leaves a connected non-singleton graph then Γ is said to be *two-connected*. Two-connectedness is equivalent to the property that every two vertices are joined by two disjoint paths, or that every pair of vertices belongs to an elementary cycle.

Let $\Gamma = (V, E)$ be a two-connected graph such that, for all $v \in V$, $\Gamma - \mathrm{star}(v)$ is connected. We say that Γ is *star two-connected*. A graph is star two-connected if and only if every two points are connected by two paths such that the distance between the paths is greater than one.

We will prove the

THEOREM 7. *If Γ is star two-connected and contains no triangle or square, then $\mathrm{aut}(F_\Gamma)$ is generated by the elementary automorphisms.*

For the graph group of a star two-connected graph Γ every locally inner automorphism is in fact inner. If Γ also contains no triangles or squares, then no vertex dominates any other vertex. To see this, let a dominate b in Γ. If a is adjacent to b then b must have valence one, otherwise there would be a triangle, but a two-connected graph has no vertices of valence one. If a is not adjacent to b, then, since b is neither isolated nor of valence one, b is adjacent to at least two vertices, say c and d, in which case $\{a, c, b, d\}$ forms a square.

So we have from theorem 7 that

THEOREM 8. *If Γ is star two-connected and contains no triangles or squares, then $out(F_\Gamma)$ is generated by the inversions and graphic automorphisms. In particular, $out(F_\Gamma)$ is finite.*

Examples of such graphs are n-gons with $n > 4$, or the graph of the dodecahedron.

LEMMA 7. *Let Γ be star two-connected and containing no triangles or squares and let ϕ be a basis for F_Γ. Then ϕ is* EN*-equivalent to a basis such that four consecutive vertices are fixed.*

Proof. Since Γ is two-connected, every two vertices lie on an elementary cycle. Take any cycle M of minimal length. M must be a full subgraph of Γ by minimality and the length of M must be greater than four. Denote any four consecutive vertices along M as follows:

Since no vertex in Γ dominates any other, we have that every vertex is in C, and, by theorem 2, we may assume that $\phi(v)$ is conjugate to v for each vertex v. Since a and b are adjacent, $\phi(a)$ commutes with $\phi(b)$ and by corollary 4, there is an element p such that $\phi(a) = pap^{-1}$ and $\phi(b) = pbp^{-1}$, so that applying an inner automorphism yields a new basis θ with $\theta(a) = a$ and $\theta(b) = b$.

Now, $\theta(c) = \mu c \mu^{-1}$, with the support of μ contained in link(b). Suppose x is an element of link(b) distinct from a and c. Then x is not adjacent any vertex in the cycle M other than b. x is not adjacent to a and c since Γ has no triangles, x is also not adjacent to the vertices in $M - b$ which are adjacent to a and c since Γ has no squares, and lastly, x is not adjacent to any other vertex of M since that would "short circuit" M, contradicting its minimality in Γ. Thus a and c are connected by the path $M - b$ in $\Gamma - \text{star}(x)$. Since $\theta(a) = a$ is not supported by x, neither is $\theta(c)$ by theorem 4, so $\text{supp}(\theta(c))$ is contained in $\{a, c\}$. This implies that $\theta(a)$ and $\theta(c)$ generated the subgroup of F_Γ generated by a and c. By theorem 5 then, $\text{supp}(\mu)$ is contained in $\{a\}$, in which case conjugating θ by a suitable power of a yields a new basis θ with $\theta(a) = a$, $\theta(b) = b$, and $\theta(c) = c$. By the same aurgument, for $\theta(d) = \delta d \delta^{-1}$, δ can only be supported by the vertex b, and conjugating θ by a suitable power of b yields a basis θ with $\theta(a) = a$, $\theta(b) = b$, $\theta(c) = c$, and $\theta(d) = d$. □

Proof. (of Theorem 6) Let ϕ be a basis. ϕ is EN-equivalent to a basis θ for which four consecutive elements of a minimal circuit M are left fixed, say the segment

We show first that $\theta(m) = m$ for all vertices m in M. Suppose to the contrary and without loss of generality that e is the other vertex adjacent to d in M and that $\theta(e) \neq e$. Then, as before, $\theta(e)$ must be supported at most by the vertex c. But this contradicts theorem 4, since c does not support $\theta(a) = a$ and a is connected to e in $\Gamma - \text{star}(c)$.

Now, Suppose that v is any vertex in Γ for which $\theta(v) \neq v$. Let $\theta(v) = \Phi v \Phi^{-1}$, and let y be a vertex in $\text{supp}(\Phi)$ distinct from v and non-adjacent to v. We have that $\Gamma - \text{star}(y)$ is connected and contains v together with at least two vertices of M, since the length of M is greater than five. But this would imply that y supports the image of both vertices, which is a contradiction. Thus $\theta(v) = v$ for all vertices □

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