

# HIGHER ORDER RIGIDITY - WHAT IS THE PROPER DEFINITION?

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ABSTRACT. We show that there is a bar-and-joint framework  $G(p)$  which has a configuration  $p$  in the plane such that the component of  $p$  in the space of all planar configurations of  $G$  has a cusp at  $p$ . At the cusp point, the mechanism  $G(p)$  turns out to be third-order rigid, in the sense that, every third-order flex must have a trivial first-order component. The existence of a third-order rigid framework that is not rigid calls into question the whole notion of higher-order rigidity.

## 1. INTRODUCTION

Suppose one is given a finite configuration of labeled points  $p = (p_1, \dots, p_n)$ , where each  $p_i$  is in Euclidean space  $\mathbb{R}^d$ . Let  $G$  be a graph whose vertices correspond to the labels  $\{1, \dots, n\}$ . An edge of  $G$ , denoted by  $\{i, j\}$ , will be called a *bar*. The configuration  $p$  together the graph  $G$  is called a *bar and joint framework*, and it is denoted by  $G(p)$ . If  $\{i, j\}$  is a bar of  $G$ , then during any continuous one parameter motion  $p(t) = (p_1(t), \dots, p_n(t))$  with  $p(0) = p$ , we will insist that the distance from  $p_i(t)$  to  $p_j(t)$  be kept fixed. When is this bar and joint framework  $G(p)$  *rigid*? That is to say, when does every such continuous motion of the points of the framework, preserving the bar lengths, arise as a restriction of a one parameter family of congruences of  $\mathbb{R}^d$ . Equivalently, when does every continuous motion preserve the distance between any two points, whether they are connected by a bar or not.

Considering each of the points  $p_i$  of the configuration as a vector in  $\mathbb{R}^d$ , the distance between  $p_i$  and  $p_j$  can be written in terms of the usual dot product, and the distance constraints are expressed by the system of quadratic equations

$$(1) \quad (p_i - p_j) \cdot (p_i - p_j) = c_{ij}, \text{ for all } \{i, j\} \text{ bars of } G,$$

where each  $c_{ij}$ , a constant, is the squared length of bar  $\{i, j\}$ . If there is an analytic motion of the points  $p_i$  satisfying (1), then all the derivatives of the left hand side must be identically zero. The resulting homogeneous differential equations provide the motivation behind the concept of first-order and second-order rigidity.

In the case of the first derivative, the definition of first-order rigidity is natural, and things work out well. Again, let  $p = (p_1, \dots, p_n)$  denote a fixed configuration corresponding to the graph  $G$ . A *first-order flex* of  $G(p)$  is a configuration of vectors  $p' = (p'_1, \dots, p'_n)$  such that the equations

$$(2) \quad (p_i - p_j) \cdot (p'_i - p'_j) = 0, \text{ for all } \{i, j\} \text{ bars of } G$$

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are satisfied. Equation 2 comes from the formal derivative of Equation 1. We say that  $p'$  is *trivial* if  $p'_i = \frac{d}{dt}\Phi_t(p_i)|_0$  for a one parameter family  $\Phi_t$ ,  $0 \leq t \leq 1$ , of congruences of  $\mathbb{R}^d$ , where  $\Phi_0$  is the identity.

If  $d = 2$ , then we may, without loss of generality, take any bar and require that bar to be fixed by any motion, essentially pinning its endpoints. Since the only one parameter family of congruences of the plane which fixes some edge is the identity, the trivial first order flexes will be easily identified by the equations  $p' = 0$ .

We say that  $G(p)$  is *first-order rigid* if it has only trivial first-order flexes. With this definition of first-order rigidity, it is easy and natural to show that if  $G(p)$  is first-order rigid, then it is rigid. See [1] and [3] for a proof and a discussion of this concept.

In the case of the second derivative, the definition of second-order rigidity is still reasonable and things still work out well. A second-order flex of  $G(p)$  is a pair of configurations of vectors  $(p', p'')$ , where  $p'$  is a first-order flex of  $G(p)$  and  $p'' = (p''_1, \dots, p''_n)$ , with  $p''_i \in \mathbb{R}^d$ ,  $i = 1, \dots, n$ , is such that the equations

$$(3) \quad (p_i - p_j) \cdot (p''_i - p''_j) + (p'_i - p'_j) \cdot (p'_i - p'_j) = 0, \text{ for all } \{i, j\} \text{ bars of } G$$

are satisfied. We also insist that  $p''_i = 0$  for all pinned vertices  $i$ . Equation 3 comes from the formal derivative of Equation 2. Physically one can regard  $p'$  as formal velocities and  $p''$  as formal accelerations permitted by the distance constraints of the framework  $G(p)$ .

We say that a bar framework  $G(p)$  is *second-order rigid* if every second-order flex  $(p', p'')$  has  $p'$  trivial as a first-order flex. In the case when  $G$  has pinned vertices, we say that  $G(p)$  is *second-order rigid* if every second-order flex  $(p', p'')$  has  $p' = 0$ . Clearly, if  $G(p)$  is first-order rigid, then it is second-order rigid. In Figure 1 we see a framework which is second-order rigid but not first-order rigid. The symbol at the end points represents a pinned vertex. In [3] it is shown that if  $G(p)$  is second-order rigid, then it is rigid.

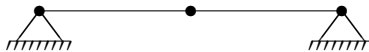


FIGURE 1. A second-order rigid framework which is not first-order rigid

Just considering the pinned case, at first sight it might seem more natural to define a framework to be second-order rigid if every second-order flex  $(p', p'')$  satisfies  $p' = p'' = 0$ . The difficulty with this definition is that it leads to a notion of second-order rigidity which is equivalent to first-order rigidity.

**PROPOSITION 1.** *A pinned framework  $G(p)$ , in the plane, is first-order rigid, if and only if every second-order flex  $(p', p'') = (0, 0)$ .*

A similar result holds in  $\mathbb{R}^d$ .

*Proof.* Suppose that  $G(p)$  is not first-order rigid. Let  $p'$  be a non-zero first-order flex of  $G(p)$  that satisfies Equation 2. Then  $(0, p')$ , where  $p'$  takes the place of a formal acceleration, is a non-zero second-order flex satisfying both Equation 2 and Equation 3.

Suppose that  $G(p)$  is first-order rigid. Let  $(p', p'')$  be any second-order flex. Since  $p'$  is a first-order flex,  $p' = 0$ . Then Equation 3 reduces to Equation 2 with  $p''$  playing the role of a first-order flex. So  $p'' = 0$  as well.  $\square$

It is thus clear that we must control only the formal first derivative in the definition of second-order rigidity. In what follows we discuss the difficulties of extending this definition to the case of higher-order rigidity. Some authors have attempted to make such definitions, but we believe that there are some inherent difficulties with any reasonable definition. See [9], pages 60-61, or [10], for example. We discuss one very natural possibility here and show the difficulties.

## 2. HIGHER-ORDER RIGIDITY

The question is how to extend these definitions of first and second-order rigidity to higher-order rigidity. The formal derivatives satisfy

$$(4) \quad \sum_{a=0}^k \binom{k}{a} (p_i^{(a)} - p_j^{(a)}) \cdot (p_i^{(k-a)} - p_j^{(k-a)}) = 0, \text{ for all } \{i, j\} \text{ bars of } G,$$

where  $p^{(a)} = (p_1^{(a)}, \dots, p_n^{(a)})$  represents the formal  $n$ 'th derivative,  $p^{(0)} = p$ , and  $\binom{k}{a}$  is the binomial coefficient. We say that  $(p^{(1)}, \dots, p^{(N)})$  is an  $N$ 'th-order flex if all the equations in (4) are satisfied for all  $k = 1, 2, \dots, N$ , and for all  $\{i, j\}$  bars of  $G$ . In [3] the following was suggested as a definition of higher-order rigidity.

**DEFINITION 1.** *A bar and joint framework  $G(p)$  is  $N$ 'th-order rigid if every  $N$ 'th-order flex  $(p^{(1)}, \dots, p^{(N)})$  has  $p^{(1)}$  trivial as a first-order flex.*

Note that this definition agrees with the definitions above for second-order rigidity and first-order rigidity, and it is clear that  $n$ 'th-order rigidity implies  $m$ 'th-order rigidity for  $1 \leq n < m$ . The hierarchy of rigidity is also non-trivial, as seen in Figure 2. (A similar figure in [3] was incorrectly described as third-order rigid.)

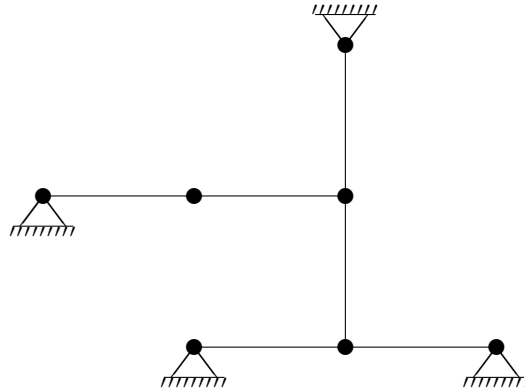


FIGURE 2. An 8'th-order rigid framework which is not 4'th-order rigid

The difficulty arises in trying to show that  $N$ 'th-order rigidity implies rigidity. If a framework has a non-trivial motion, which we may assume, without loss of generality, to be analytic (see [3] for a proof of this analytic parametrization), then it would at first appear that we can take as many derivatives as needed to distinguish the motion from a trivial one. Those derivatives will be a non-trivial flex of some order. So the following question arises. Are there *mechanisms* (with pinned vertices), that is, non-rigid frameworks, all of whose non-trivial flexes have a trivial first-order component? The mechanism must at least have this property

to be a candidate for being an example of being  $N$ 'th order rigid, but not rigid. Regard each configuration  $p$  as a single vector in  $\mathbb{R}^{nd}$  and consider the configuration space  $X$ ,

$$X = \{p \mid p \text{ is a configuration satisfying (1)}\} \subset \mathbb{R}^{nd},$$

where pinned vertices are fixed in the definition of  $X$ . A motion of a mechanism corresponds to a path in  $X$ , and so a curve in  $\mathbb{R}^{nd}$ . In this general context, if a curve has a cusp (such as the curve given by  $y^2 = x^3$  at the point  $(0,0)$  in the plane), then the velocity of any analytic parametrization at the cusp point must be 0. These considerations motivate the following example.

### 3. THE EXAMPLE

In Figure 3 we have an example of a mechanism whose configuration space  $X$  has a cusp at this position. We will see that it is in fact third-order rigid by Definition 1. The framework basically is made up of two mirror image Watt mechanisms joined

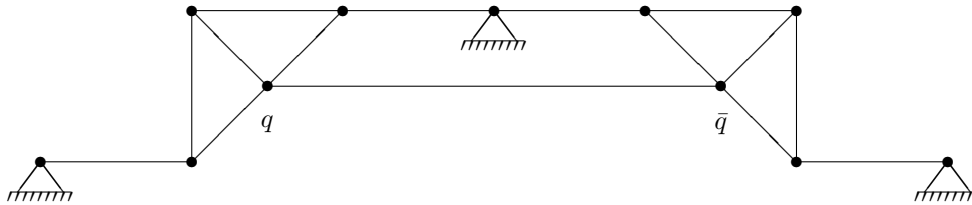


FIGURE 3. A third-order rigid framework that is a mechanism

by a horizontal rod. For a discussion of the Watt mechanism see [8]. A Watt mechanism is pictured in Figure 4, where the path of the midpoint  $q$  of the middle

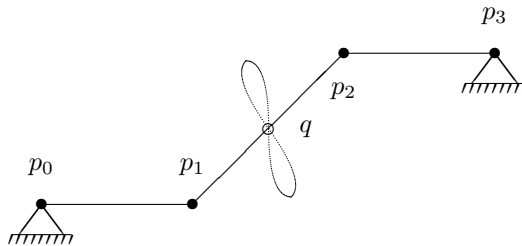


FIGURE 4. The Watt mechanism

bar under the motion of the framework is indicated. Note that, although there is a self-intersection of this path, from the configuration pictured, the motion of  $q$  must follow first along the arc having the vertical tangent. With some additional bars added to allow the attachment of an edge at this midpoint, it follows that the central horizontal bar of Figure 3 is confined to the well of Figure 5a, from which position it cannot ascend, and may descend in two alternative ways shown in Figure 5b and Figure 5c, depending on which endpoint falls faster. It is clear that infinitesimally the two paths away from this configuration coincide, hence the path in the configuration space has a cusp at the position of Figure 3.

**PROPOSITION 2.** *The framework in Figure 3 is third-order rigid but not rigid.*

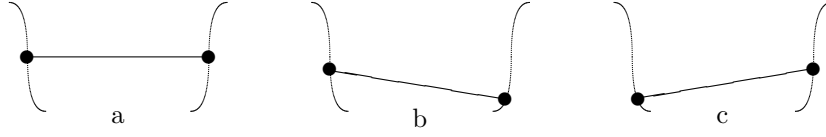


FIGURE 5. Possible configurations near the cusp

*Proof.* The point  $p_1$  in Figure 4 is constrained by the bar  $\{0, 1\}$  to move in a circle about  $p_0$ , so the most general third order flex at  $p_1$  satisfies

$$p'_1 = (0, a_1), \quad p''_1 = (-a_1^2, b_1), \quad \text{and} \quad p'''_1 = (-3a_1b_1, c_1).$$

Similarly, the most general third-order flex at  $p_2$  satisfies

$$p'_2 = (0, a_2), \quad p''_2 = (a_2^2, b_2), \quad \text{and} \quad p'''_2 = (3a_2b_2, c_2),$$

and the bar  $\{1, 2\}$ , with  $p_2 - p_1 = (1, 1)$ , gives three equations;

$$(1, 1) \cdot (0, a_2 - a_1) = 0, \quad (1, 1) \cdot (2a^2, b_2 - b_1) = 0,$$

$$(1, 1) \cdot (3a(b_2 + b_1), c_2 - c_1) = 0,$$

where  $a_1 = a_2 = a$  from the first equation. Thus

$$b_2 - b_1 = -2a^2, \quad c_2 - c_1 = -3a(b_2 + b_1).$$

It follows that, all the non-pinned vertices on the left augmented Watt mechanism of Figure 3 will have  $(0, a)$  as their first-order flex. Moreover, since all these vertices are part of an infinitesimally rigid framework, all the higher order  $p^{(a)}$ 's are determined by  $p_1^{(a)}$  and  $p_2^{(a)}$ . See [3]. In particular, the point  $q$  in Figure 4 will have its most general third-order flex  $(q', q'', q''')$  expressible as the arithmetic mean of  $(p'_1, p''_1, p'''_1)$  and  $(p'_2, p''_2, p'''_2)$ . So we may write,

$$q' = (0, a), \quad q'' = (0, (b_2 + b_1)/2), \quad \text{and} \quad q''' = (-6a^3, (c_2 + c_1)/2).$$

By symmetry,

$$\bar{q}' = (0, \bar{a}), \quad \bar{q}'' = (0, (\bar{b}_2 + \bar{b}_1)/2), \quad \text{and} \quad \bar{q}''' = (6\bar{a}^3, (\bar{c}_2 + \bar{c}_1)/2).$$

The central horizontal bar, with, say,  $\bar{q} - q = (2, 0)$  yields three equations. The first-order equation is already satisfied. The second-order equation is

$$(2, 0) \cdot (0, (\bar{b}_2 + \bar{b}_1 - b_2 - b_1)/2) + (0, \bar{a} - a) \cdot (0, \bar{a} - a) = 0$$

giving  $a = \bar{a}$ , and hence  $(\bar{q}' - q') = 0$ . Thus the third-order equation gives

$$(2, 0) \cdot (12a^3, (\bar{c}_2 + \bar{c}_1 - c_2 - c_1)/2) = 0,$$

so  $a = 0$ , and the first-order flex  $p' = 0$ .  $\square$

#### 4. COMMENTS

Much of what has been discussed here can be put in a more general context. For example, the definition of  $N$ 'th-order rigidity can be made in the context of arbitrary algebraic sets. It also follows that first and second-order rigidity implies rigidity. But of course it does not hold for  $N$ 'th-order rigidity for  $N \geq 3$ .

It also should be true that any local singularity is equivalent to the neighborhood of a point in the configuration space  $X$  in  $\mathbb{R}^{nd}$  of an appropriate mechanism. The notion of equivalence should be chosen so that the non-zero terms in an analytic parametrization should be preserved. This is related to the well-known result of

Kempe [8] that any plane curve has a mechanism that has one of its vertices trace out any given compact portion of the curve defined by polynomial inequalities say. See [6] for a careful proof. The problem is to do this in the configuration space, and not just for one vertex.

The problem still remains as to what kind of definition one might use for “higher-order rigidity”. The following properties seem to us to be necessary for any reasonable definition:

1. First and second order rigidity agree with Definition 1.
2. For  $N = 1, 2, \dots$ , if a framework is  $N$ 'th order rigid, then it is  $(N + 1)$  order rigid, (but not conversly).
3. For  $N = 1, 2, \dots$ , if a framework is  $N$ 'th order rigid, then it is rigid.

We just saw that Definition 1 does not satisfy Property 3. But it does satisfy the others and the following additional desirable properties as well.

4. If a given framework is rigid, then it is  $N$ 'th order rigid for some  $N = 1, 2, \dots$
5. For  $N = 1, 2, \dots$ , for a given framework,  $N$ 'th order rigidity can be computed in a finite time.

One idea to strengthen the previous definitions of higher order rigidity is to use Proposition 1 as the model. One may require, say, that some fixed fraction of any flex be trivial. In fact, second-order rigidity is equivalent to the condition that every flex of even order  $(p^{(1)}, \dots, p^{(2N)})$  has  $(p^{(1)}, \dots, p^{(N)})$  trivial, as was shown in [3] in the course of proving that second-order rigidity implies rigidity. Proceeding in this manner, one obtains a heirarcy of weaker definitions, and it can be shown that the first four conditions above are all satisfied. Unfortunately, we do not yet know if the necessary finiteness requirement of condition 5 can be achieved for  $N > 3$ .

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