PERMUTATION REPRESENTATIONS OF THE SYMMETRY GROUPS OF REGULAR HYPERBOLIC TESSELLATIONS

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ABSTRACT. Higman has questioned which discrete hyperbolic groups $[p, q]$ have representations onto almost all symmetric and alternating groups. We call this property $\mathcal{H}$ and show that, except perhaps for finitely many values of p and q, $[p, q]$ has property $\mathcal{H}$.

It is well known that the modular group $\Gamma = \langle x, y \mid x^2 = y^3 = 1\rangle$ has the property that every alternating and symmetric group is a homomorphic image of Γ except A_6, A_7, A_8, S_5, S_7, or S_8 [5]. Higman has questioned which discrete reflective hyperbolic groups also exhibit this type of behavior. Let G be an infinite, finitely presented group. We say that G has *property* $\mathcal{H}$ if there is an integer $N > 1$ such that either A_n or S_n is a homomorphic image of G for all $n > N$. If p and q satisfy $(p-2)(q-2) > 4$, then $\{p, q\}$ denotes the tessellation of the hyperbolic plane by regular p–gons, q meeting at each vertex, and $[p, q]$ denotes the symmetry group of that tessellation. It is an infinite Coxeter group generated by the reflections in the sides of the right hyperbolic triangle forming the fundamental region of the tessellation, and has the presentation

$$[p, q] = \langle R_1, R_2, R_3 | R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^2 = (R_2R_3)^p = (R_1R_3)^q = 1\rangle.$$

Let us assume without loss of generality that $p \leq q$. For $[p, q]$ we will actually use the presentation

$$[p, q] = \langle x, y, t | x^2 = y^p = (xy)^q = (t)^2 = (xt)^2 = (yt)^2 = 1\rangle, \tag{1}$$

with the correspondence given by $x = R_1R_2$, $y = R_2R_3$ and $t = R_2$, which exhibits $[p, q]$ as a semi–direct product

$$[p, q] = \langle x, y | x^2 = y^p = (xy)^q = 1\rangle \times \langle t | t^2 = 1\rangle.$$

Geometrically, x and y are rotations at the vertices of the fundamental region having angles $\pi/2$ and π/p respectively, and t is the reflection on the side joining them. $[p, q]^+$ denotes the index 2 subgroup of orientation preserving isometries in $[p, q]$. For more details see [4].

It has been shown that $[3, q]$ has property $\mathcal{H}$ for all $q > 6$, see [2], and that $[4, q]$ has property $\mathcal{H}$ for all $q > 6$, see [6]. We will show that $[p, q]$ has property $\mathcal{H}$ for all for all but perhaps finitely many values of p and q. Presumably, $[p, q]$ has property $\mathcal{H}$ for all values of p and q, and a careful examination of any particular case has so far led to the conclusion that it has property $\mathcal{H}$. In view of the results cited above, and since $[p, q]$ is a homomorphic image of $[pr, q]$, it is enough to show this for p prime, $p > 3$. In particular, we will show that

THEOREM 1. *If $q > 57$, then either A_n or S_n is a homomorphic image of $[5, q]$ for all $n > 4(57)(58)$.*

THEOREM 2. *If p is prime, $p \geq 7$, $q > 3p+2$, then either A_n or S_n is a homomorphic image of $[p,q]$ for all $n > (2q)(2q+1)$.*

THEOREM 3. *If p is prime, $25 \leq p \leq q$, then either A_n or S_n is a homomorphic image of $[p,q]$ for all $n > [(2q)-16][(2q+1)-16]$.*

Suppose $[p,q]$ has property $\mathcal{H}$, $p \leq q$, and define $N_{p,q}$ to be the largest value of n such that A_n or S_n is not a homomorphic image of $[p,q]$.

COROLLARY 1. $N_{p,q} = O(q^2)$.

1. $\mathcal{H}$–GRAPHS

We will prove Theorems 1, 2 and 3 by exhibiting finite graphs on whose vertex sets the group $[p,q]$ acts as the full permutation group. Equivalently, this graph can be interpreted as a coset diagram, with the vertices identified with the cosets of $A = \mathrm{Stab}(v)$, the stabilizer of some vertex v of the graph, or as the 1–skeleton of the cover of the fundamental complex of the presentation which corresponds to the subgroup A.

The fundamental 2–complex of a presentation is formed by taking a single vertex, attaching an oriented loop for each generator, and sewing in a 2–cell for each relation. The fundamental group of the complex therefore realizes the group, and given any subgroup, the corresponding cover gives a complex upon which the quotient acts freely and transitively. In particular, it acts on the 2–skeleton, which will be an oriented colored graph, the color of an edge being determined by the generator to which it corresponds. Using the presentation in 1, we will refer edges as being x–edges, y–edges, and t–edges, rather than, say, red, blue and green.

Following suggestions of Higman, as was done in [1, 2, 6], we will reduce the complexity of a coset diagram D for $[p,q]$ while retaining the action of $[p,q]$ on the vertex set. Since x and t are involutions, it is appropriate to indicate their action by unoriented edges. Loops, indicating fixed points by a generator, will not be pictured. Because of the semi-direct product structure, the action of t induces a color preserving, orientation reversing graph automorphism of the diagram H obtained by deleting the t–edges. Since D is connected, H has either 1 or 2 components. We will assume that H has one component, and will indicate the action of t not by edges, but by drawing the diagram in the plane with a vertical axis of symmetry and following the convention that all y cycles are oriented counter-clockwise. We will call such a diagram, together with its embedding, an $\mathcal{H}$–graph. The fixed points of y are the pendant vertices, the fixed points of x are the vertices of valence 2, and the fixed points of t are those that lie on the symmetry axis.

For a given n, we wish to construct $\mathcal{H}$–graphs with n vertices such that $[p,q]$ acts as the full symmetric group. The creation of planar $\mathcal{H}$–graphs can be easily accomplished, as in the following proposition.

PROPOSITION 1. *Let Γ be a connected bicolored graph embeded in the plane with an axis of symmetry. Γ is an $\mathcal{H}$–graph corresponding to a subgroup of the group $[p,q]$ if*

1. *The the subgraph induced by the y–edges of Γ is a disjoint union of k–gons, each k dividing p, and each k–gon bounds a face.*
2. *the subgraph induced by the x–edges of Γ is a disjoint union of edges.*

(3) *if F is a face, including the exterior face, whose whose boundary is not a k–gon of y–edges, then the number of pendant vertices plus the number of y–edges along the boundary divides q.*

2. Orbits in $\mathcal{H}$–graphs

Given an $\mathcal{H}$–graph, we are interested in the size of the orbits of the element $xyt = R_1R_2R_3$, which is of infinite order in $[p, q]$ since it acts as a translation on a Petrie polygon of the tessellation, see [4].

A non–pendant x–edge in an $\mathcal{H}$–graph is a *bridge* if it joins two different y–cycles, and a *chord* otherwise.

A vertex v in a subgraph S of an $\mathcal{H}$–graph is a *vertex of attachment* if v is incident to an edge not in S, the other vertices of S will be referred to as *interior vertices*. An edge of the subgraph which is incident to a vertex of attachment is called an *edge of attachment*. A *fragment*, F, is a connected induced subgraph of an $\mathcal{H}$–graph such that F has exactly two vertices of attachment, all edges of attachment are y–edges joining a vertex of valence 2 to the vertex of attachment, and $t(F) \cap F$ is either empty or equal to F.

A *comb* is a fragment of an $\mathcal{H}$–graph both whose y–edges form a path joining the vertices of attachment, such that there is a pendant x–edge at every vertex except the endpoints of the edges of attachment. A *caravan* is a fragment of an

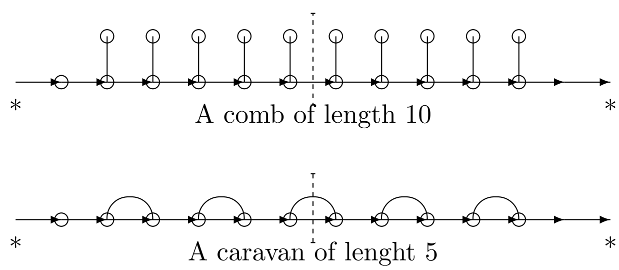

FIGURE 1.

$\mathcal{H}$–graph whose y–edges form a path joining the vertices of attachment, such that, except for the endpoints of the edges of attachment, each vertex along this path is connected to one of its neighbors by an x–edge. A comb and a carvan together with their mirror images are pictured in Figures 1 and 2 with the vertices of attachment indicated by a *, and the reader can readily verify that the circled vertices comprise an orbit by the element xyt, giving the following:

LEMMA 1. *Let F be a comb in an $\mathcal{H}$–graph with j pendant x–edges. Then, except for the vertex of valence 2 at the end of the path of y–edges, all the other vertices in the fragment belong to a single orbit of xyt of size $4j + 2$ if $t(F) \cap F = \emptyset$, or of size $2j + 1$ if $t(F) = F$.*

LEMMA 2. *Let F be a caravan in an $\mathcal{H}$–graph with j x–edges. Then, except for the vertex of valence 2 at the end of the path of y–edges, all the other vertices in the fragment belong to a single orbit of xyt of size $2j + 2$ if $t(F) \cap F = \emptyset$, or of size $j + 1$ if $t(F) = F$.*

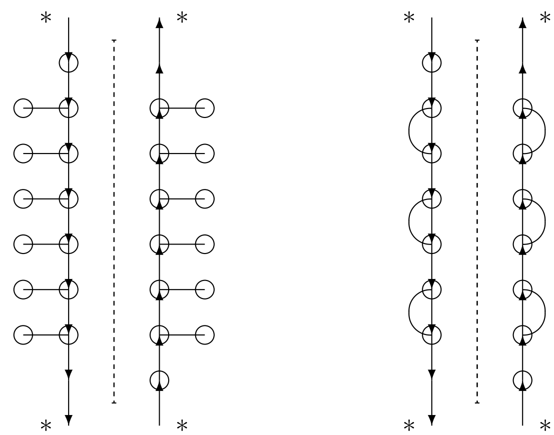

FIGURE 2.

By chasing the orbit of xyt in the fragments in Figures 1 and 2, one sees that the fixed points of x cause the orbit of xyt to turn back on itself. Generally, if there are enough such fixed points in a fragment, the orbits by xyt cannot propagate, and their size will be bounded. We say a fragment F is *sparse* if its interior vertices satisfy

(1) a pendent x–edge which in not part of a comb is distance 1 away from a bridge, and distance 1 away from a vertex of valence 2,
(2) no y–edge connects two bridges, and no bridge touches the axis of symmetry,
(3) every chord is part of a carvan.

We note that were we not also concerned with the case $p = 5$, we could strengthen [3] to say that every pendant x–edge is part of a comb, but this would leave a pentagon connected by two bridges with no room for any pendant edges at all, thus we allow, for example, the situation in Figure 3.

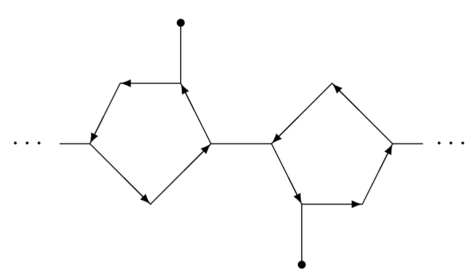

FIGURE 3.

LEMMA 3. *Let F be a sparse fragment of an $\mathcal{H}$–graph Γ, and consider the action of the element xyt. If both a and ay are interior vertices which are not both part of a comb or a carvan, then $a^{xyt} \subseteq F \cup t(F)$, and $|a^{xyt}| \in \{2, 3, 4, 8, 12\}$.*

Proof. Let v and vy be interior vertices, and suppose first that they are distinct. If they are both of degree 2, then the situation is pictured in Figure 4, depending on the placement of the axis of symmetry, (the vertices in the orbit are indicated by a large circle, and the vertex serving the role of v is marked with v or v').

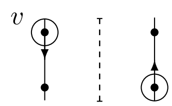

FIGURE 4.

If v is of degree 2, and vy is of degree three with a pendent x–edge, then vyy is not a vertex of attachment either. The vertex vyy has valence 3, otherwise v and vy are part of a (short) comb, so the x–edge at vyy is a bridge and $vyyx$ is an interior vertex.

If $vyyxy$ has valence 2, the situation is depicted in Figure 5a. If $vyyxy$ is of

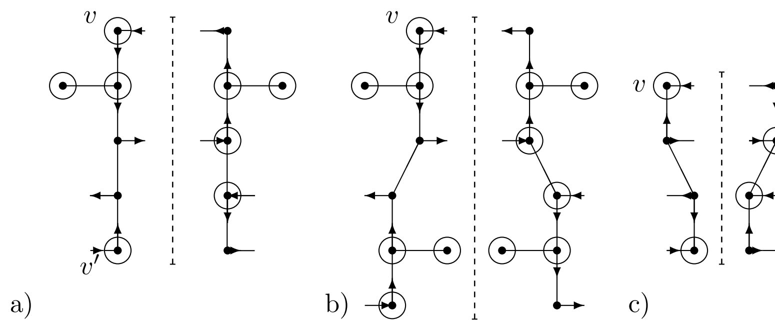

FIGURE 5.

valence 3 then the x–edge at $vyyxy$ must be pendant and the situation is pictured in Figure 5b. If v has valence 2 and vy has valence 3, then the situation is either as pictured in Figure 5c or looks like that of the orbit of v' in Figure 5a.

If v is of valence 1, and v is not part of a comb, then the situation can also be found among the circled orbits listed in Figure 5. □

3. Fundamental Orbits

If group G acts on a set S, then let $s_S^g = s^g$ denote the orbit of $s \in S$ under the action of $g \in G$. A *block* is a subset $B \subseteq S$ such that $gB \cap B$ is either B or $\emptyset$. The action of G on S is *primitive* if the only blocks are S, $\emptyset$ and the singleton sets.

The orbit s^g is a *fundamental orbit* if $|s^g|$ is prime, the size of no other orbit is divisible by $|s^g|$, and $(s^g)a \cap s^g \neq \emptyset$ for all a in some generating set A for G. To show that a group acts as the symmetric or alternating group is tantamount to finding a fundamental orbit, which is the content of the following proposition, whose proof follows an argument of Higman.

PROPOSITION 2. *Let G act on S, $|S| = n$, with fundamental orbit s^g with $|S - s^g| \geq 3$, then either $G/S = S_n$ or $G/S = A_n$.*

Proof. We first show that the action is primitive.

The orbits of S by g partition S, and since $|s^g|$ is coprime to the size of all the other orbits, some power $h = g^i$ cycles the elements of s^g while fixing all other elements of S, so $s^g = s^h$. Let B be a non-empty block which, by transitivity, we may assume intersects s^h. If there is an element $b \in B$ with $b \notin s^h$, then, since h fixes b and cycles the elements of $s^g = s^h$, $s^h \subseteq B$. Since $(s^h)a \cap s^h \neq \emptyset$ for all $a \in A$, $Ba = B$ for all $a \in A$, and $B = S$.

If, on the other hand, $B \subseteq s^h$, then since $|s^h|$ is prime, either $B = s^g$, and again, by the above argument, $B = s^g = S$, or every B is a singleton. Thus the action is primitive.

Since the action is primitive, and since h has degree and order a prime p with $|S| - p \geq 3$, the result follows by a theorem of Jordan, see [8], Theorem 13.9. □

4. The Handle Construction

An *α–handle* in an $\mathcal{H}$–graph is a vertex v of valence 2 such that $vt = v(xy)^\alpha$, so v is an α–handle if and only if vt is a $(|v^{xy}| - \alpha)$–handle. If v is an α–handle, then the path $\{v, vx, vxy, vxyx, \ldots\}$ bounds a face which is cut by the symmetry axis of the $\mathcal{H}$–graph.

Given two $\mathcal{H}$–graphs U and V, with α–handles u and v respectively, we may construct a new $\mathcal{H}$–graph, $U +_{u,v} V$ by making the $\{v, vx, vxy, vxyx, \ldots\}$ face the exterior face of V, and placing the graph of V in the $\{u, ux, uxy, uxyx, \ldots\}$ face of U, joining u to v and ut to vt by x–edges. The same effect may be had by stacking V and U one above the other and adding the new x–edges from u to v and from ut to vt. It is well known, see [7], and straightforward to verify, that $U +_{u,v} V$ is also an $\mathcal{H}$–graph satisfying the following.

Proposition 3. *If $(u)_U^{(xyt)} \cap (ut)_U^{(xyt)} = \emptyset$ and $v_V^{(xyt)} \cap (ut)_V^{(xyt)} = \emptyset$, then $u_{(U+_{u,v}V)}^{(xyt)} = u_U^{(xyt)} \cup v_V^{(xyt)}$, and $(ut)_{(U+_{u,v}V)}^{(xyt)} = (ut)_U^{(xyt)} \cup (vt)_V^{(xyt)}$.*

Two $\mathcal{H}$–graphs $\mathcal{B}$ and $\mathcal{S}$ form a *brother and sister pair* if there are α–handles b_α and s_α and β–handles b_β and s_β in $\mathcal{B}$ and $\mathcal{S}$ respectively so that

1. the orbits $(b_\alpha)^{(xyt)}$, $(b_\alpha t)^{(xyt)}$, $(b_\beta)^{(xyt)}$, and $(b_\beta t)^{(xyt)}$ are all pairwise disjoint, as well as $(s_\alpha)^{(xyt)}$, $(s_\alpha t)^{(xyt)}$, $(s_\beta)^{(xyt)}$, and $(s_\beta t)^{(xyt)}$,
2. $|(b_\alpha)^{(xyt)}| + |(s_\alpha)^{(xyt)}|$ is a prime not dividing any of the values $\{|v^{(xyt)}| : v \in \mathcal{B}\}$, $|w^{(xyt)}| : w \in \mathcal{S}\}$, $|(b_\alpha t)^{(xyt)}| + |(s_\alpha t)^{(xyt)}|$, $|(b_\beta)^{(xyt)}| + |(s_\beta)^{(xyt)}|$, and $|(b_\beta t)^{(xyt)}| + |(s_\beta t)^{(xyt)}|$.
3. Either $(b_\alpha)^{(xyt)}$ or $(s_\alpha)^{(xyt)}$ contains two vertices joined by a y–edge,
4. Either $(b_\alpha)^{(xyt)}$ or $(s_\alpha)^{(xyt)}$ contains two vertices symmetric in t.

Theorem 4. *If $\mathcal{B}$ and $\mathcal{S}$ are a brother and sister pair of $\mathcal{H}$–graphs and $|\mathcal{B}|$ and $|\mathcal{S}|$ are coprime, then either A_n or S_n is a homomorphic image of $[p, q]$ for all $n > |\mathcal{B}||\mathcal{S}|$.*

Proof. Let $F_{i,j}$ denote the $\mathcal{H}$–graph

$$\cdots +_{b_\beta} \mathcal{B} +_{b_\alpha} \mathcal{B} +_{b_\beta} \mathcal{B} +_{b_\alpha} \mathcal{S} +_{b_\beta} \mathcal{S} +_{b_\alpha} \mathcal{S} +_{b_\beta} \cdots,$$

with i copies of $\mathcal{B}$ and j copies of $\mathcal{S}$. Since $|\mathcal{B}|$ and $|\mathcal{S}|$ are coprime, by suitably choosing i and j, the number of vertices in $F_{i,j}$ can take on any value greater than $|\mathcal{B}||\mathcal{S}|$. The orbit of xyt formed at h_α when the top copy of $\mathcal{B}$ is joined to the bottom copy of $\mathcal{S}$ is a fundamental orbit for the action, and the result follows. □

As an example, consider the brother and sister pair in Figure 6. The fundamental orbit is indicated by the circled vertices. This example shows that $[5, 17]$ has $S_{(24)(25)}$ as a homomorphic image.

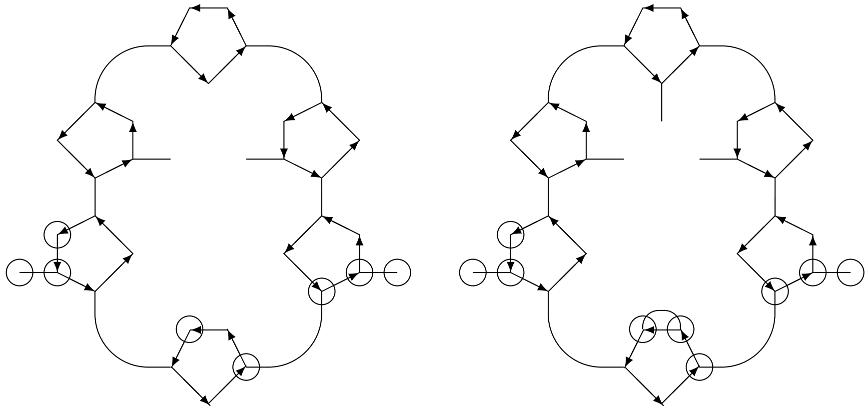

FIGURE 6.

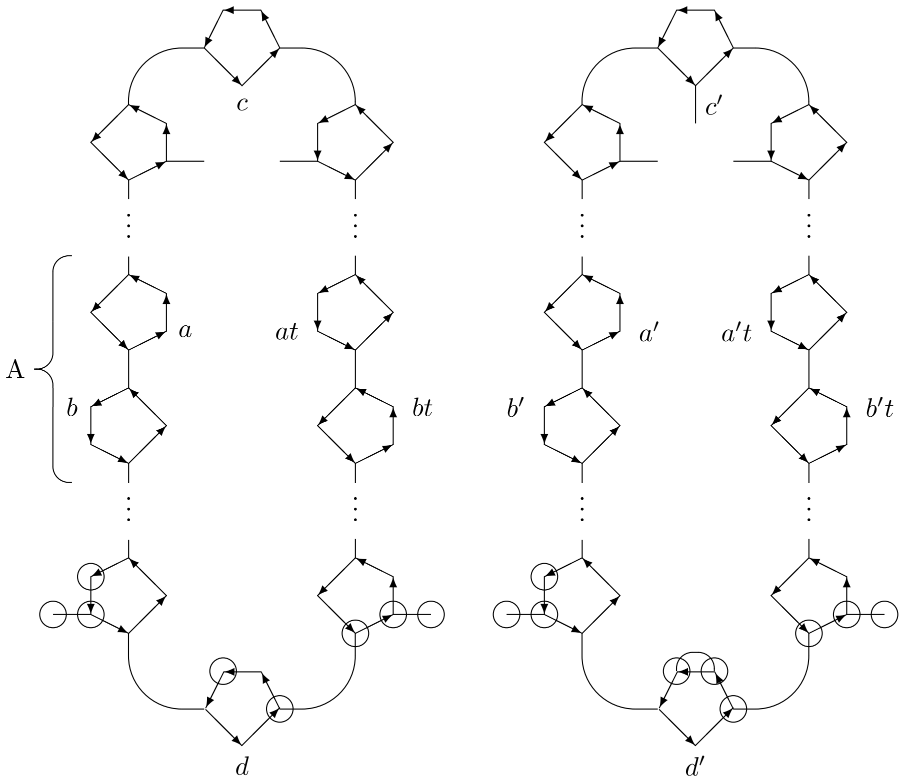

FIGURE 7.

5. THE CASE $p = 5$

More generally, consider the pair in Figure 7. In this figure, the section marked A can occur several times, and if there are i copies then these are $\mathcal{H}$–graphs for $[5, 17 + 10i]$. Attaching a pendant x–edge at a, at, a' and $a't$ will change $\mathcal{B}$ and

$\mathcal{S}$ from being an $\mathcal{H}$–graph for $[p, q]$ to one for $[p, q + 2]$, just as adding pendant x–edges at c_1 c_2, and c_2', while removing the pendant x–edge at c_1', will change $\mathcal{B}$ and $\mathcal{S}$ from being an $\mathcal{H}$–graph for $[p, q]$ to one for $[p, q+1]$. Thus, if $i \geq 4$ we have enough room to add pendant x–edges so that q can take any value $q \geq 17+40 = 57$. For any such choice, $\mathcal{B}$ and $\mathcal{S}$ are both sparse except for the top pentagon, and the bottom three pentagons. It is easy to check that the orbits there are all coprime to 17, and so these form a brother sister pair for $[p, q]$ for all $q \geq 57$, proving Theorem 1.

6. The case $p \geq 7$

For $p \geq 7$, we consider a similar pair, depicted in Figure 8, which are a brother

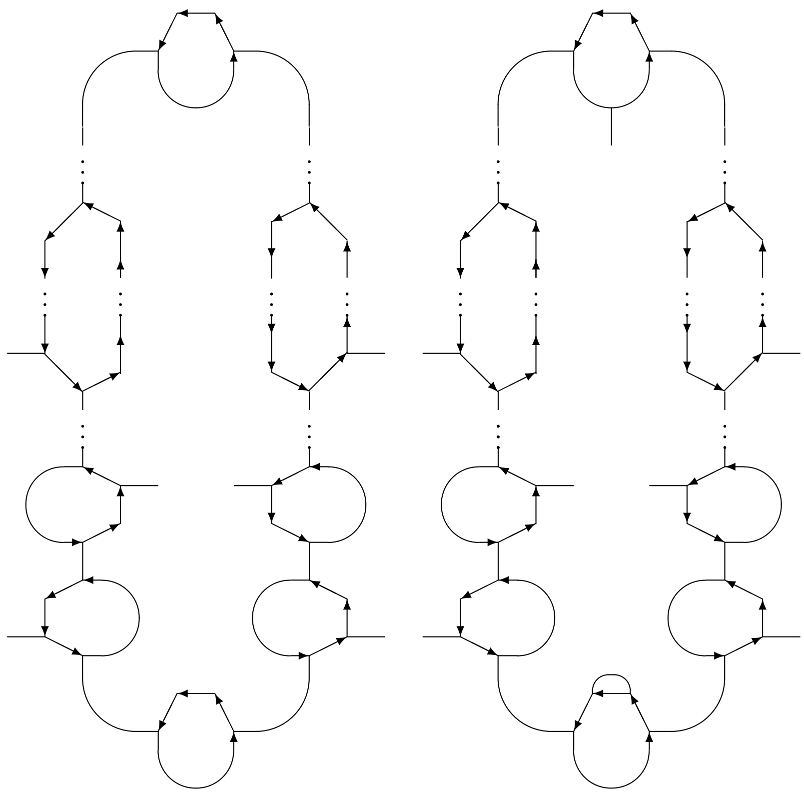

Figure 8.

sister pair for $[p, q]$ with $q \equiv (3p+2) \bmod (p+1)$. On each of the 6 long y–paths in the non–repeated section, we can add a total of $p - 6$ pendant x–edges without disturbing the sparsity of the fragments, and still leaving enough room to shift a pendant x–edge away from a comb which contributes an orbit of xyt whose size is a multiple of 17. This allows us to increase the size of q by any value up to $3(p - 6)$, and $3(p - 6) \geq p$ when $p \geq 9$, so if $9 \leq p$ and $3p + 2 \leq q$, then A_n or S_n is a homomorphic image of $[p, q]$ whenever $ngeq(6p+2)(6p+3)$. We get the same result for $p = 7$, since it is possible to add 2 pendant edges sparsely to the long arcs of the septagons. This proves Theorem 2.

7. The case $p \geq 25$

In the previous cases, we were able to augment uniformly the size of the orbits of xy, and hence q, by adding x–edges to form combs of controlled length. If p is large enough then we can also reduce q by adding x–edges in the form of a caravans alongs the long paths of y–edges, each link of which introduces a fixed point of xy, and decreases some orbit by 1. Let $p \geq 25$ and consider the pair in Figure 9.

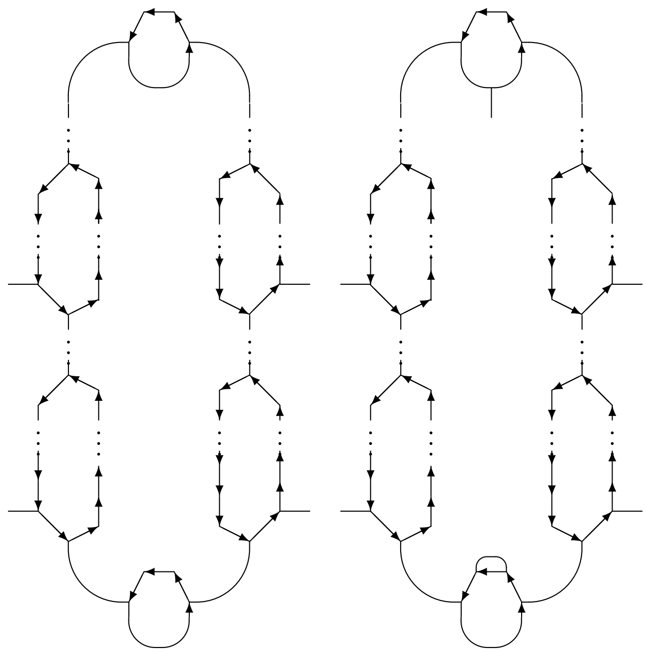

Figure 9.

Using the same arguments as in the previous section, we see that, with the addition of some sparse pendant x–edges, these form a brother sister pair for $[p, q]$ for any $q \geq 2p + 1$.

Similarly, the pair in Figure 10. is a brother sister pair for $p \leq q \leq 2p - 16$.

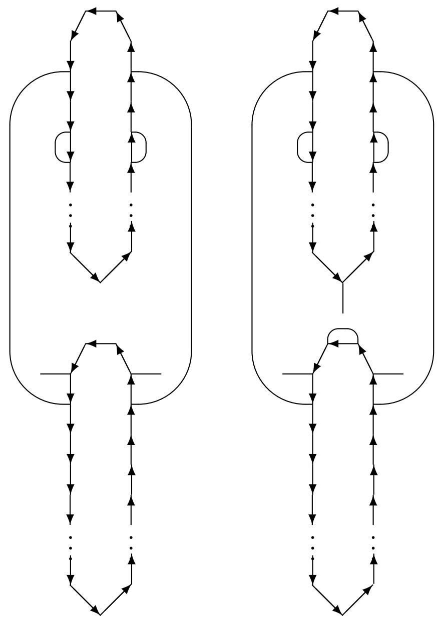

Figure 10.

The values $2p - 15 \leq q \leq 2p$ can be attained by adding sufficient caravans to the the smallest version of the pair in Figure 9. To achieve this range, there must be room for caravans with at least eight x–edges on each y–polygon, which is assured as soon as $p \geq 25$. This proves Theorem 3.

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