

NUMBER THEORY - TAKE HOME TEST - FEBRUARY 2009

Please do your choice of four problems. Write them up neatly and carefully. You may use your notes and the textbook. Do not use any other sources. Do not collaborate with anyone in or out of the class. If you have questions, or you think there is an error in a problem, please email me.

**Problem 1.** The number  $100 = 10^2$  has 9 positive divisors,  $\{1, 2, 4, 5, 10, 20, 25, 50, 100\}$ . Show that for any number  $n$ ,  $n^2$  has an odd number of positive divisors.

**Problem 2.** Suppose that  $n^2$  is odd and that 3 does not divide  $n^2$ . Show that  $12 \mid (n^2 - 1)$ .

**Problem 3.** Suppose  $a, b, c$  and  $d$  are all integers greater than 1. Let  $a$  be coprime to  $b$ ,  $c$  coprime to  $d$  and  $ab$  be coprime to  $cd$ . Must it be true that  $ac$  is coprime to  $bd$ ? If so prove it is true, if not give a counterexample.

**Problem 4.** Find an ideal  $I \subseteq \mathbb{Z}$  such that  $504 \in I$ , and  $1080 \in I$  and  $17,340 \in I$ , but  $-40 \notin I$ .

**Problem 5.** Let  $a$  and  $b$  be integers. Show that if  $\gcd(a, b) \neq 1$  then there is a number  $n \geq 2$  such that  $n^2 \mid ab$ .

**Problem 6.** Let  $a, b, c$  and  $d$  be integers such that  $ab$  is coprime to  $cd$ . Show that  $\gcd(a, b)$  is coprime to  $\text{lcm}(c, d)$ .

**Problem 7.** Show that  $\gcd(\gcd(a, b), c) = \gcd(\gcd(a, c), b)$

**Problem 8.** Show that  $\text{lcm}(\text{lcm}(a, b), c) = \text{lcm}(\text{lcm}(a, c), b)$

**Problem 9.** Let  $n$  be an integer. Show that  $3 \mid n^6 - n^2 + 3$ .