GENERIC AND ABSTRACT RIGIDITY

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Rigidity

We are all familiar with frameworks of rods attached at joints. A rod and joint framework gives rise to a simple mathematical model consisting of line segments in Euclidean 3-space with common endpoints. A *deformation* is a continuous one-parameter family of such frameworks. If a framework has only trivial deformations, e.g. translations and rotations, then it is said to be *rigid.* Before giving a more precise mathematical formulation, we can use simple geometry to explore these ideas.

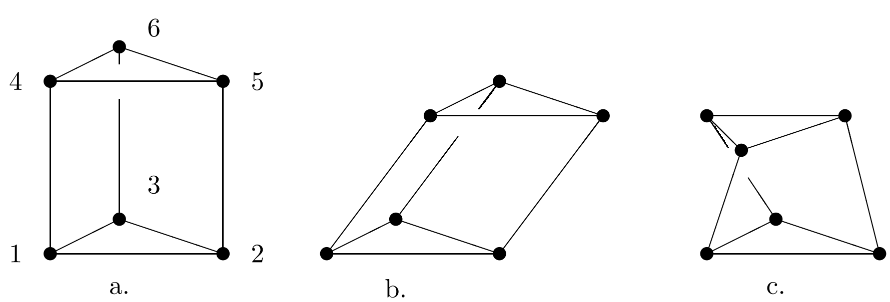


Figure 1: Deformations of the triangular prism in 3D.

Consider the triangular prism of Figure 1a. It has an obvious deformation in which the bottom triangle is held fixed and the three posts rotate simultaneously about their lower endpoints, remaining parallel throughout. As the posts move, any two of them form the sides of a parallelogram, and so the triangles formed by their upper points are congruent, see Figure 1b. Another deformation is to keep the planes of the two triangles parallel and "screw down" the top triangle, as in Figure 1c. We can try to roughly count the deformations by piecing together the framework. Starting with the bottom triangle, 123, without loss of generality we may assume that it is fixed, and then we need not concern ourselves with trivial deformations. Adding segment 14, we must have that point 4 is constrained to move on a sphere, which has two degrees of freedom. For any position of point 4, adding segments 25 and 45 constrains point 5 to lie on the intersection of two spheres, i.e. a circle, with one degree of freedom. Lastly we add segments 36, 46 and 56, so that point 6 is constrained to lie on the intersection of three spheres, and so is fixed by what has already been chosen. Altogether we have three degrees of freedom for deformations of the triangular prism. That this is a rough calculation can be seen from Figure 2, which shows a "flattened"

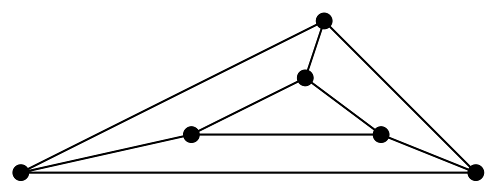


Figure 2: A rigid prism.

triangular prism which is rigid because the inner triangle is held tightly in place by the spokes. Notice that the rigidity of the flattened prism is unstable in the sense that, if the framework is not precisely in the plane, then a deformation is possible.

It is a different question entirely to consider a framework which is constrained to move only in the plane. Beyond being of theoretical interest, such frameworks arise in special engineering applications, as well as geometric questions arising from computer aided design, [12]. Regarding the Figure 1a as a drawing of a plane framework, it also has an obvious deformation. Again let us count the degrees of freedom. As before let the bottom triangle be fixed. Adding segment 14 constrains 4 to move on a circle, which has one degree of freedom. Adding segments 25 and 45 constrains 5 to move on the intersection of two circles, which is a single point. The same with adding edges 36 and 46. Thus the framework with one segment missing has only one degree of freedom. Adding the last edge constraint should remove that last degree of freedom and yield a framework which is rigid in the plane. So in the plane we expect a triangular prism framework to be rigid, but there nevertheless exist flexible ones, while in 3-space we expect a triangular prism framework to be flexible, yet there exist rigid ones.

We now make precise what we mean by a framework. A *graph* (V, E) consists of a vertex set $V = \{1, 2, \ldots, n\}$ and edge set E, where E is a collection of unordered pairs of vertices. A *framework* is a triple $(V, E, \mathbf{p})$ where (V, E) is a graph and $\mathbf{p} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ is a list of distinct points of m dimensional Euclidean space corresponding to the vertices of V.

If $\{i, j\}$ is an edge of (V, E), then a *deformation* of the framework is a continuous one parameter family $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t))$ with $\mathbf{p}(0) = \mathbf{p}$, such that the distance from $\mathbf{p}_i(t)$ to $\mathbf{p}_j(t)$ is kept fixed if $\{i, j\} \in E$,

$$(\mathbf{p}_i(t) - \mathbf{p}_j(t)) \cdot (\mathbf{p}_i(t) - \mathbf{p}_j(t)) = c_{ij}, \text{ for all } \{i, j\} \text{ edges of } G, \quad (1)$$

The framework $(V, E, \mathbf{p})$ is said to be *rigid* if all deformations are locally trivial, that is, $\mathbf{p}(t)$ is congruent to $\mathbf{p}$ for all t near 0. Equivalently, a deformation is trivial if it preserves the distance between any two points, whether they are adjacent or not.

Finding deformations, or even solving the system of equations 1 is very difficult in general, and we have already seen some delicate special cases. One successful approach is to not to look for deformations directly but to look for their first derivatives. If $\mathbf{p}$ is a framework and $\mathbf{p}(t)$ is a deformation, then, by Equation 1, its derivative $d\mathbf{p}/dt = \mathbf{p}'$ must satisfy

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = \mathbf{0}, \text{ for all } \{i, j\} \in E \quad (2)$$

For $t = 0$ this is a system of $|E|$ linear equations with the nm unknowns being the coordinates of the $\mathbf{p}'_i$. If, as we shall hereafter assume, the framework contains at least $m + 1$ points in general position, then there are always $m(m + 1)/2$ solutions of Equations 2 corresponding to the derivatives of trivial deformations. If the system of equations (2) has no other solutions, then we say the framework $(V, E, \mathbf{p})$ is *first order rigid*, or *infinitesimally rigid*. A solution to Equations 2 is called an *infinitesimal flex* or just a *flex*. Infinitesimal rigidity is a natural approximation to rigidity, and the connection between them must be carefully examined.

If $(V, E, \mathbf{p})$ is infinitesimally rigid, then any deformation must have its initial velocities coinciding with that of a trivial deformation, or, in other words, any deformation which fixes enough vertices to prevent trivial deformations must have initial velocity 0 at every vertex. If every deformation could be parameterized so that the initial velocities were non-zero, then it would be obvious that infinitesimal rigidity implies rigidity, however, in 1992 the bar and joint framework of Figure 3 was discovered which is a cusp point in its configuration space [1]. The embedding $\mathbf{p}$ of a framework $(V, E, \mathbf{p})$ may be regarded as defining a single point of mn dimensional space, and the subspace of $\mathbb{R}^{mn}$ comprising all solutions of the equations (1) is called the *configuration space of* $(V, E, \mathbf{p})$. Pinning sufficiently many vertices to prevent trivial deformations, indicated in Figure 3 by the grounds, a framework with a one dimensional configuration space is called a *mechanism*. A framework which is a cusp mechanism must have all deformations with initial velocity 0. (The actual motion of the cusp

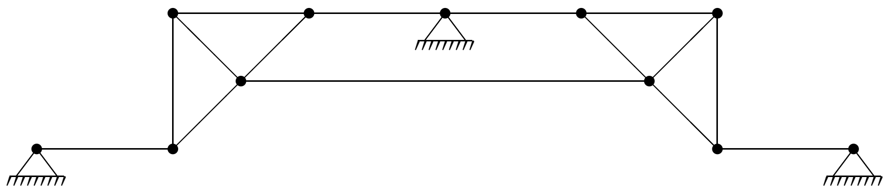


Figure 3: A third order rigid framework that is a mechanism

framework of Figure 3 can currently be viewed at http://www.wpi.edu/˜ hservat/index.html.) Nevertheless, any deformation $\mathbf{p}(t)$, which can be assumed to be analytic, must have some derivative which is non-zero at $t = 0$, and it can be shown that the lowest order derivative of $\mathbf{p}(t)$ which is non-zero at $t = 0$ satisfies Equation 2, which means the framework is not infinitesimally rigid and we have the following theorem.

THEOREM 1 *Infinitesimal rigidity implies rigidity.*

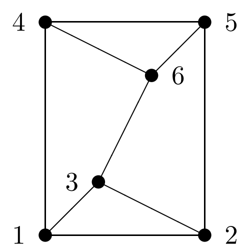


Figure 4: An infinitesimally rigid triangular prism in 2D.

EXAMPLE: The 2D framework of Figure 4 is infinitesimally rigid. If we have a flex, then, using isometries, we may assume that triangle 456 is fixed and $\mathbf{p}'_4 = \mathbf{p}'_5 = \mathbf{p}'_6 = 0$. Equation 2 applied to bars $\{1,4\}$ and $\{2,5\}$ implies that $\mathbf{p}'_1$ and $\mathbf{p}'_2$ are both horizontal. Then, since bar $\{1,2\}$ is horizontal, $(\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{p}'_1 - \mathbf{p}'_2) = 0$ implies $\mathbf{p}_1 = \mathbf{p}_2$. Therefore, using the constraints of bars $\{1,3\}$ and $\{2,3\}$ we have that $\mathbf{p}'_3 - \mathbf{p}'_1 = \mathbf{p}'_3 - \mathbf{p}'_2$ is simultaneously perpendicular to bars $\{1,3\}$ and $\{2,3\}$, hence $\mathbf{p}'_1 = \mathbf{p}'_2 = \mathbf{p}'_3$, all horizontal. But the constraint that $\mathbf{p}'_3 - \mathbf{p}'_6 = \mathbf{p}'_3$ be perpendicular to bar $\{3,6\}$ implies that $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_3 = 0$ and the flex is trivial. □

The converse of Theorem 1 is not true – if a framework is rigid, it may nevertheless have an infinitesimal flex.

EXAMPLE: The flattened triangular prism of Figure 5 is rigid in 3-space, but has an infinitesimal flex which assigns zero velocity to the vertices of the outer triangle, and velocities to the inner triangle, all of which are perpendicular to the plane of the prism. □

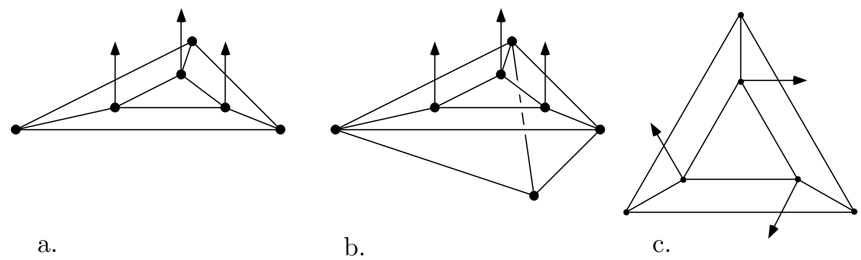


Figure 5: Flexes on a flat prism.

Actually, any framework in 3-space all of whose vertices lie in a plane has many flexes of the type of Figure 5a, since each equation of (2) is equivalent to

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot \mathbf{p}_i' = (\mathbf{p}_i - \mathbf{p}_j) \cdot \mathbf{p}_j'$$

which says geometrically that the initial velocities of the endpoints of any bar have equal projections in the direction of the bar, as in Figure 6. This geometric

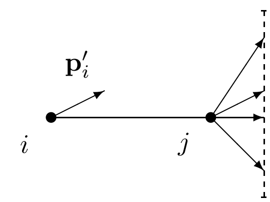


Figure 6: The infinitesimal edge condition.

interpretation is very useful in the construction of small examples.

Example: The framework of Figure 5a violates our assumption that a framework in m-space has at least $m + 1$ points in general position, so we augment the flat prism in 3D by another vertex as in Figure 5b in which case the flex implies that the framework of Figure 5b is not infinitesimally rigid in 3D.

If the flattened prism is regarded as a 2D framework, then the flex of Figure 5a is not valid, however, there is still the infinitesimal flex of Figure 5c, which fixes the outer triangle and is an infinitesimal rotation of the inner triangle. □

Example: Another example of a rigid framework with an infinitesimal flex is the 2D embedding of the triangular prism illustrated in Figure 7a, which, despite the flex indicated by the arrows in Figure 7b, is rigid. Holding the lower triangle fixed, the outer equal-length vertical bars force every point of the upper triangle to move in a circular path whose radius is equal to the length of those bars. The middle vertical bar however forces the lower

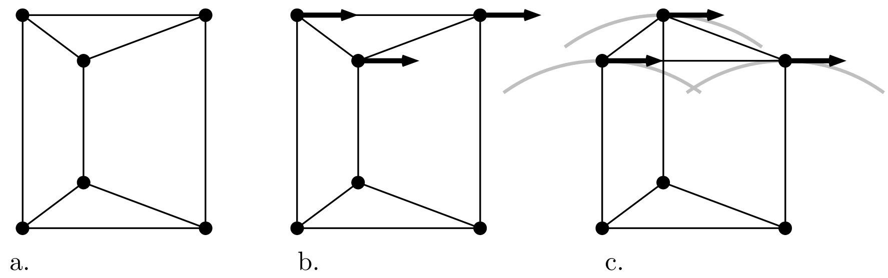


Figure 7: An infinitesimally flexible, rigid prism, and a prism mechanism.

point to move in a circle of strictly smaller radius and so the framework is rigid in the plane. The framework of Figure 7c has an infinitesimal flex which comes from a deformation. □

One can generalize the idea of infinitesimal rigidity to try to detect rigid but not infinitesimally rigid frameworks. The most successful is to look for an infinitesimal flex which is compatible with an assignment of formal initial accelerations, $\mathbf{p}_i''$. Taking the derivative of the equations in (2) we have

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i'' - \mathbf{p}_j'') + (\mathbf{p}_i' - \mathbf{p}_j') \cdot (\mathbf{p}_i' - \mathbf{p}_j') = 0, \text{ for all } \{i, j\} \in E \qquad (3)$$

which are taken to be the constraints on $\mathbf{p}_i''$. If, given a framework $(V, E, \mathbf{p})$, there exists vector assignments to the vertices $\mathbf{p}'$ and $\mathbf{p}''$ such that $\mathbf{p}'$ is a flex and $\mathbf{p}'$ and $\mathbf{p}''$ satisfy the equations in (3), then we say the framework has a *second order flex.* A framework all of whose second order flexes have trivial first order flexes is said to be *second order rigid*, and it can be shown that second order rigidity implies the rigidity of the framework, [2].

Example: Consider the flat prism in 2D pictured in Figure 5c. Assuming without loss of generality that $\mathbf{p}'$ and $\mathbf{p}''$ are zero on the outer triangle, it is easy to calculate that the only solution to the system of equations (2) and (3) has $\mathbf{p}'$ zero on all vertices and $\mathbf{p}''$ as pictured by the arrows of Figure 5c, so the framework is 2'nd order rigid but not infinitesimally rigid. □

It is natural to hope, therefore, that a hierarchy of higher order rigidities could be developed such that, given any rigid framework, it is k'th order rigid for some k. This can be done, but it is not effective for detecting rigidity since, for example, the cusp framework of Figure 3, which is not rigid, is 3'rd order rigid.

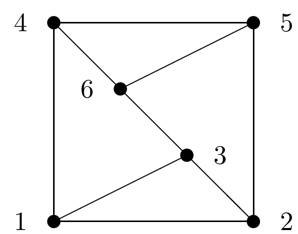
In general, we can regard frameworks which are rigid but not infinitesimally rigid as being in some sense singular, and we see from the examples that the flexes, while not being realizable in an ideal motion of the framework, will certainly give rise to sagging or swaying in any physical model of bars and joints. Thus, for civil engineering applications, infinitesimal rigidity is a superior concept both in terms of the applicability as well as the computational complexity.

The Rigidity Matrix

Given a framework $(V, E, \mathbf{p})$, determining if it is infinitesimally rigid involves solving the system of equations (2), which is a system of $|E|$ linear equations in nm unknowns. Naturally we can form the matrix of this system, $R(V, E, \mathbf{p})$, whose rows correspond to the edges and whose columns correspond to the coordinates of the vertices. Since $V = \{1, 2, \ldots, n\}$, we order the coordinates of the vertices as

$$\{(\mathbf{p}_1)_1, \ldots, (\mathbf{p}_1)_m, (\mathbf{p}_2)_1, \ldots, (\mathbf{p}_2)_m, \ldots (\mathbf{p}_n)_1, \ldots, (\mathbf{p}_n)_m\}$$

and order the edges lexicographically. Then the row of the matrix corresponding to edge $\{i, j\}$ has the coordinates of the vector $(\mathbf{p}_i - \mathbf{p}_j)$ in the columns from $im + 1$ to $im + m$, $(\mathbf{p}_j - \mathbf{p}_i)$ in the columns from $jm + 1$ to $jm + m$, and zero elsewhere. $R(V, E, \mathbf{p})$ is an $|E| \times nm$ matrix.



$\mathbf{p}_1 = (0, 0)$ $\quad$ $\mathbf{p}_4 = (0, 3)$

$\mathbf{p}_2 = (3, 0)$ $\quad$ $\mathbf{p}_5 = (3, 3)$

$\mathbf{p}_3 = (2, 1)$ $\quad$ $\mathbf{p}_6 = (1, 2)$

Figure 8: An embedding of a 2D prism.

EXAMPLE: Consider the 2D framework of Figure 8. The rigidity matrix is:

$$R(V, E, \mathbf{p}) = \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 2 & 1 \end{bmatrix}$$

and it is not difficult, owing to the the large number of zero entries, to show that this matrix is of rank $9 = 2|V| - 3$, hence its framework is infinitesimally rigid, hence rigid. $\square$

Since the rigidity matrix is large even for small examples, it is common to use a more compact notation by assigning m-dimensional vector entries to an

$|E| \times n$ matrix, so that, e.g., $R(V, E, \mathbf{p})$ for a triangular prism is:

$$\begin{bmatrix} (\mathbf{p}_1 - \mathbf{p}_2) & (\mathbf{p}_2 - \mathbf{p}_1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (\mathbf{p}_1 - \mathbf{p}_3) & \mathbf{0} & (\mathbf{p}_3 - \mathbf{p}_3) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (\mathbf{p}_1 - \mathbf{p}_4) & \mathbf{0} & \mathbf{0} & (\mathbf{p}_4 - \mathbf{p}_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{p}_2 - \mathbf{p}_3) & (\mathbf{p}_3 - \mathbf{p}_2) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{p}_2 - \mathbf{p}_5) & \mathbf{0} & \mathbf{0} & (\mathbf{p}_5 - \mathbf{p}_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{p}_3 - \mathbf{p}_6) & \mathbf{0} & \mathbf{0} & (\mathbf{p}_6 - \mathbf{p}_3) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{p}_4 - \mathbf{p}_5) & (\mathbf{p}_5 - \mathbf{p}_4) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{p}_4 - \mathbf{p}_6) & \mathbf{0} & (\mathbf{p}_6 - \mathbf{p}_4) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{p}_5 - \mathbf{p}_6) & (\mathbf{p}_6 - \mathbf{p}_4) \end{bmatrix}$$

Setting $\mathbf{q}_{ij} = (\mathbf{p}_i - \mathbf{p}_j)$ we have the even more compact form:

$$\begin{bmatrix} \mathbf{q}_{12} & \mathbf{q}_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{q}_{13} & \mathbf{0} & \mathbf{q}_{31} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{q}_{14} & \mathbf{0} & \mathbf{0} & \mathbf{q}_{41} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{q}_{23} & \mathbf{q}_{32} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{q}_{25} & \mathbf{0} & \mathbf{0} & \mathbf{q}_{52} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{q}_{36} & \mathbf{0} & \mathbf{0} & \mathbf{q}_{63} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{q}_{45} & \mathbf{q}_{54} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{q}_{46} & \mathbf{0} & \mathbf{q}_{64} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{q}_{56} & \mathbf{q}_{65} \end{bmatrix}$$

In this last form we see a similarity between $R(V, E, \mathbf{p})$ and the vertex edge adjacency matrix of the graph (V, E). In fact, if $m = 1$, then the rows of $R(V, E, \mathbf{p})$ are constant multiples of the corresponding rows of the adjacency matrix of (V, E). This similarity will be exploited in the sections on generic rigidity.

If $(V, E, \mathbf{p})$ has an infinitesimal flex, then that flex is a solution to Equation 2 and so, written as a column vector, is an element of the nullspace of $R(V, E, \mathbf{p})$. Since we are assuming that every framework has at least $m+1$ points in general position, the trivial deformations constitute a subspace of dimension $m(m+1)/2$, so the graph is infinitesimally rigid if that is the dimension of the kernel of $R(V, E, \mathbf{p})$.

THEOREM 2 *A framework on n vertices is infinitesimally rigid in m-space if and only if its rigidity matrix has rank $mn - \frac{m(m+1)}{2}$.*

We can use all the tools of linear algebra to determine infinitesimal rigidity. For example, since the row and column rank of a matrix are the same, one way to compute the rank of a matrix is to compute its cokernel - that is, the subspace of vectors $\mathbf{u}$ with $\mathbf{u}R(V, E, \mathbf{p}) = \mathbf{0}$. These row vectors correspond to an assignment of scalars u_{ij} to the edges of the framework so that, at each vertex i, we have

$$\sum u_{ij}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0},$$

where the sum is taken over all j adjacent to i. Such a $\mathbf{u}$ is called a *resolvable stress* on the framework, and may be interpreted as an assignment of spring constants to the edges such that the resulting force at each vertex is $\mathbf{0}$, leaving the framework at equilibrium. Since a resolvable stress is a row dependency of the rigidity matrix, the existence of a resolvable stress is equivalent to the condition that one of the edges of the framework may be deleted without affecting its infinitesimal rigidity.

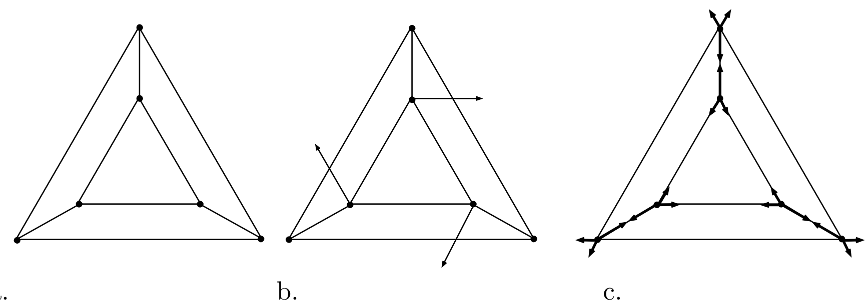


Figure 9: A deformation and a stress on a 2D Triangular Prism.

EXAMPLE: Although the 2D triangular prism of Figure 9a has enough edges to be infinitesimally rigid, we have seen that it has a flex. So the rank of $R(V, E, \mathbf{p})$ is strictly less than $6 \cdot 2 - 3 = 9$ and hence there must be a dependence among the edges. This stress is illustrated in Figure 9c, in which the stress on an edge is indicated by two arrows at its endpoints which point inward if the stress is positive, and outward if it is negative. □

Historically, the the study of stresses, resolvable and unresolvable, came before the study of infinitesimal rigidity and began with the work of Maxwell, see [9]. See also [2] for a modern treatment.

Limiting Frameworks

Practically, one may object that the embeddings we are considering allow edges to cross. This may present difficulties, especially in the plane, if we are building physical frameworks. If our applications involve virtual frameworks, however, then it may be essential to allow such crossing. For example, in computer aided design, uniquely specifying a 3D rendering of a framework involves rigidity, see [12]. Obviously, specifying the 2D rendering on a computer screen of a 3D framework involves 2D rigidity of frameworks with crossing edges. In fact, such rendered frameworks can have vertices which coincide, and theoretically, we should not exclude this case as well.

If we allow vertices to coincide, nothing in the previous pages is altered with respect to either rigidity or infinitesimal rigidity if the vertices which occupy the same location in $\mathbb{R}^m$ are non-adjacent. Allowing adjacent vertices to coincide is more problematic. In ordinary rigidity, if two adjacent vertices coincide, then the constraint between those vertices states that the distance between them is zero, so those two vertices together have only two degrees of freedom, whereas two vertices joined by an edge of positive length have three degrees of freedom. In infinitesimal rigidity, the situation is reversed. If $\mathbf{p}_1 = \mathbf{p}_2$, then $(\mathbf{p}_1 - \mathbf{p}_2)\cdot(\mathbf{p}'_1 - \mathbf{p}'_2) = 0$, the infinitesimal condition on $\mathbf{p}'_1$ and $\mathbf{p}'_2$, is automatically satisfied, so, infinitesimally, two vertices joined by an edge of length zero have four degrees of freedom, instead of three, and the row of the rigidity matrix corresponding to that edge is zero.

A more interesting and useful concept is the limit of frameworks. Specifically, we are interested in the case where one point $\mathbf{p}_i$ approaches another point $\mathbf{p}_j$ along a direction defined by unit vector $\mathbf{q}$, see Figure 10 where the directions of the unit vectors $\mathbf{q}$ are indicated by the directions of the arrowheads. Of course, the limit of the rigidity matrices will have a zero row corresponding to edge $\{i, j\}$. To avoid this we define the *normalized rigidity matrix*, $NR(V, E, \mathbf{p})$, which is obtained from the ordinary rigidity matrix by dividing each of the rows by the norm of the vectors in that row, i.e., if we write $NR(V, E, \mathbf{p})$ in the compact form then every non-zero entry is a unit vector.

It is easy to see that if $\mathbf{p}$ is a limit framework with $\mathbf{p}_a$ approaching $\mathbf{p}_b$ along direction $\mathbf{q}$, then $NR(V, E, \mathbf{p})$ is the matrix whose $\{i, j\}$'th row is $(\mathbf{p}_i - \mathbf{p}_j)/|\mathbf{p}_i - \mathbf{p}_j|$ in columns $m(i-1)+1$ to mi and $(\mathbf{p}_j - \mathbf{p}_i)/|\mathbf{p}_i - \mathbf{p}_j|$ in columns $m(j-1)+1$ to mj if i is adjacent to j and $\{i, j\} \neq \{a, b\}$ and $\mathbf{q}$ and $-\mathbf{q}$ in columns $m(a-1)+1$ to ma and $m(b-1)+1$ to mb respectively.

If the limit framework has a set of edges which is *independent*, that is, it corresponds to a set of independent rows in the rigidity matrix, then the continuity of the determinant function implies that there exists a nearby ordinary framework whose corresponding edges are also independent.

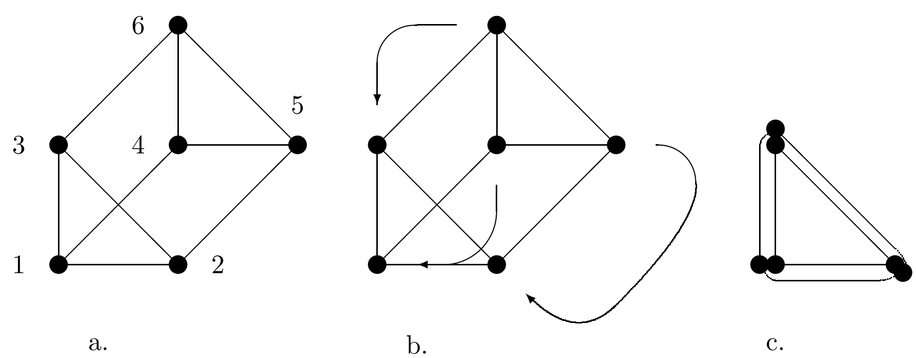


Figure 10: The limit of a framework.

EXAMPLE: The framework of Figure 10a with $\mathbf{p}_1 = (0,0)$, $\mathbf{p}_2 = (1,0)$, $\mathbf{p}_3 = (0,1)$, $\mathbf{p}_4 = (1,1)$, $\mathbf{p}_5 = (2,1)$, and $\mathbf{p}_6 = (1,2)$, is another flexible 2D prism, so, since it has 9 edges, there must be a dependence among the edges, and every 9×9 minor of its rigidity matrix is zero.

$$R(V,E,\mathbf{p}) = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

If we look at the limit as $\mathbf{p}_1 \to \mathbf{p}_4$, $\mathbf{p}_2 \to \mathbf{p}_5$, and $\mathbf{p}_3 \to \mathbf{p}_6$ along the paths indicated Figure 10b, then we get the limiting framework indicated in Figure 10c, which has normalized rigidity matrix

$$NR(V,E,\lim \mathbf{p}) = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & -\alpha & -\alpha & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & \alpha & 0 & 0 & 0 & 0 & \alpha & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & -\alpha & -\alpha & \alpha \end{bmatrix}$$

with $\alpha = \sqrt{2}/2$. Since the matrix has rank 9, we conclude that there exists an ordinary embedding of the prism in 2-space whose edges are independent. We will see in the section on generic rigidity in the plane a nice way to prove that this matrix is of full rank. □

Generic Rigidity

Clearly, analyzing the rigidity and the infinitesimal rigidity of a framework requires one to study both the incidences of the segments as well as their placement in space. In combinatorial rigidity, [5], we try to separate these aspects and study to what extent can we judge the rigidity of a framework merely from knowing the number of points and segments and their incidences, in other words, just from knowing the underlying graph of the framework.

If we wish to avoid singular frameworks, then we want to avoid dependencies that arise from only from the embedding. We say that a given framework $(V, E, \mathbf{p})$ is *generic* if all frameworks sufficiently near $\mathbf{p}$ are have the same infinitesimal rigidity properties as $(V, E, \mathbf{p})$. (When we speak of the distance between frameworks, it is as points in $\mathbb{R}^{nm}$.) An embedding $\mathbf{p}$ is said to be *generic* if each framework $(V, E, \mathbf{p})$ is generic, for any graph (V, E) on V. It is not hard to show that almost all embeddings in m-space are generic [5]. In fact, it can be shown that the non-generic embeddings form an algebraic set of codimension greater than 2 in $\mathbb{R}^{nm}$, so the generic embeddings form an open connected dense subset of $\mathbb{R}^{nm}$, and hence two generic frameworks with the same graph are either both infinitesimally rigid or they are both not infinitesimally rigid. Thus we may define a *graph* (V, E) to be *generically rigid* in dimension m if it has a generic embedding in m-space which is rigid.

We also say that a framework is generically rigid if its graph is generically rigid, however it should be noted that this is only a statement about the graph of the framework. In other words, a generically rigid framework may be a non-generic framework, and may be non-rigid.

Example: The graph of the triangular prism, which has been the leitmotiv of this article, is generically rigid in dimension 1. In one dimension, the normalized rigidity matrix is just the incidence matrix of the graph and, in fact, the concepts of rigidity, infinitesimal rigidity, and generic rigidity all coincide. □

From the argument of the previous example, we have that a graph is generically rigid in dimension 1 if and only if it is connected, and that every framework in dimension 1 is generic.

Example: The graph of the triangular prism is generically rigid in dimension 2. The 2 dimensional rigidity matrix of a triangular prism is 9 by 12. We have previously seen that the triangular prism has an infinitesimally rigid embedding in the plane, see Figure 4. This embedding has only trivial flexes, and the kernel of the rigidity matrix has dimension 3, and so its rank is $12 - 3 = 9$, which implies that the rigidity matrix has a 9 by 9 submatrix of non-zero determinant. Since generic embeddings are an open dense subset of m-space, and since the determinant is a continuous function, there must be a nearby generic embedding of the triangular prism for which the corresponding submatrix of its rigidity matrix is also non-zero. Hence the rigidity matrix of the generic framework is also of rank 9, and so the generic framework is infinitesimally rigid Thus the graph of the triangular prism is generically rigid in the plane. □

The argument of the previous example proves the following.

Theorem 3 *An m dimensional framework which is infinitesimally rigid is also generically rigid.*

A similar argument shows the following useful tool.

Theorem 4 *An m dimensional framework which has an infinitesimally rigid limit framework is generically rigid.*

Example: Since the graph of the triangular prism is generically rigid in the plane, every framework on that graph is generically rigid in the plane. However, it is possible to embed a prism in the plane so that it is not infinitesimally rigid, see Figure 7a, or even so that it is deformable, see Figure 7c. □

EXAMPLE: The graph of the triangular prism is not generically rigid in 3-space. We have seen that it has an embedding into 3-space which is rigid, however that embedding was not infinitesimally rigid. An infinitesimally rigid embedding into 3-space would allow only trivial flexes, so the rigidity matrix should have a kernel of dimension 6 and have rank $3 \cdot 6 - 6 = 12$, which is impossible since it has only 9 rows. □

THEOREM 5 *A graph with n vertices which is generically rigid in dimension m has at least $m \cdot n - \frac{m(m+1)}{2}$ edges.*

The relationship between rigidity, infinitesimal rigidity and generic rigidity is summarized in the Venn diagram of Figure 11, where the frameworks inside the

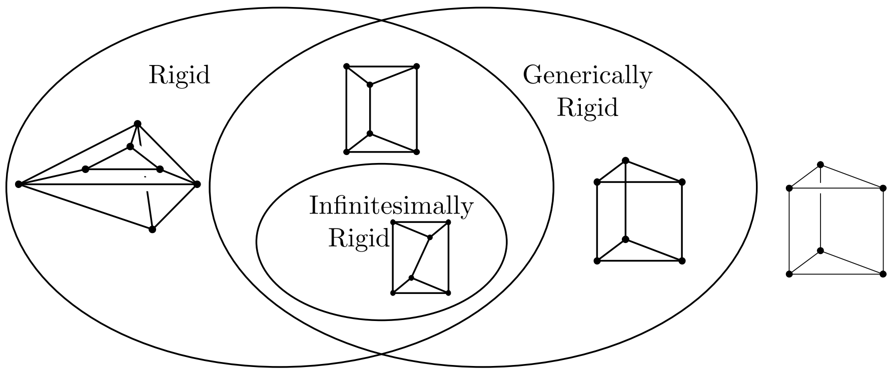


Figure 11: The rigidity map.

generic region are 2D, and those outside are 3D. Since generic rigidity depends on the graph alone, it is natural to look for purely graph theoretic, or at least combinatorial, characterizations. In dimension 1 we have seen that generic rigidity is equivalent to connectivity. We will see in the next section that there is a completely analogous development in dimension 2 which simultaneously makes two dimensional rigidity accessible via graph theory, and gives rise to new theoretical tools for graph theory itself. In dimension three and higher, such a characterization has so far eluded the mathematicians.

Generic Rigidity in the Plane

Theorems 3 and 5 imply that a graph $G = (V, E)$ which is generically rigid in the plane must have at least $2n - 3$ independent edges, that is, $2n - 3$ edges which span the vertices and are distributed "wisely". If there are more than $2k - 3$ edges connecting a subset of k vertices, the corresponding rows of the rigidity matrix must be dependent. Thus, in order for a graph to be generically rigid in the plane, there must be a subset $F \subseteq E$ satisfying the following two conditions:
[**L1:**] $|F| = 2n - 3$

[**L2:**] For all $F' \subseteq F$, $F' \neq \emptyset$, $|F'| \leq 2k - 3$, where k is the number of vertices which are endpoints of edges in F'.

That these two conditions are also sufficient was proved by Laman in 1970, [7].

THEOREM 6 (Laman's Theorem)$G = (V, E)$ *is generically rigid in the plane if and only if there is a subset F of E satisfying* **L1** *and* **L2**.

Laman's Theorem is a combinatorial characterization of generic rigidity in the plane. We can use it to determine if a graph is generically rigid without embedding the vertices into $\mathbb{R}^2$ and calculating the rank of the rigidity matrix. Unfortunately, Laman's conditions are not easy to check for large graphs because we are required to check all subsets $F \subseteq E$ of the correct cardinality, and then all subsets thereof. A direct algorithmic approach to generic rigidity using Laman's Theorem would clearly be exponential in the number of vertices. However, there is a striking similarity between Laman's Theorem and the following theorem of Nash-Williams [10].

THEOREM 7 (Nash-Williams) *A graph is the edge disjoint union of two spanning trees if and only if $|E| = 2n - 2$ and, for all $E' \subseteq E$, $|E'| \leq 2k - 2$, where k is the cardinality of the set of endpoints of E'.*

Edmonds [4] provided a polynomial time algorithm for decomposing a graph into two spanning trees and we can easily get a polynomial algorithm for testing a graph for generic rigidity: For each set $F \subseteq E$, $|F| = 2n - 3$, Edmonds' algorithm yields the union of two trees after adding any one edge to F. Recski [11] showed that it is enough to double every edge of F, which decreases the number of times Edmonds' algorithm needs to be applied. Crapo [3] observed that the rigid subset we are looking for is actually the union of three trees such that every vertex is contained in exactly two of them, and such that no two non-trivial subtrees have the same underlying vertex set. Crapo calls such a decomposition a *proper* 3T2 *decomposition.* It is easy to see the connection between Nash-Williams' Theorem and Crapo's 3T2 decomposition: Remove one edge of a graph which is the union of two spanning trees, and you have a 3T2 decomposition of the remaining graph (not necessarily proper, since two subtrees may span the same set of vertices.)

Crapo's algorithm [3, 13] always produces a 3T2 decomposition such that one of the trees is spanning, however there are many 3T2 decompositions where all three trees have approximately the same size. It is an open question if Crapo's algorithm can be altered to yield such a balanced decomposition, which would improve the running time.

Tay in [13] showed directly, without using Laman's Theorem, that Crapo's condition characterizes generic rigidity in the plane and we sketch his proof here.

THEOREM 8 (Tay) *A graph $G = (V, E)$ is generically rigid and independent in the plane if and only if it has a proper 3T2 decomposition.*

PROOF: Assume G has $2n-3$ edges and there exists a generic embedding $\mathbf{p}$ of V into $\mathbb{R}^2$ such that $R(G)$ has rank $2n-3$. Then there exists a $(2n-3) \times (2n-3)$

submatrix R' of R with $\mathrm{Det}(R') \neq 0$, in fact R' can be taken to be R with the last three columns deleted (the last two columns are clearly always dependent on the previous ones). Expanding the determinant along the odd columns, we can find submatrices R_1 and R_2 of R', R_1 an $(n-1)\times(n-1)$ matrix consisting of the odd columns of R' and rows corresponding to the edge set E_1 and R_2 an $(n-2)\times(n-2)$ matrix consisting of the even columns of R' and rows corresponding to edge set $E_2 = E - E_1$, such that $\mathrm{Det}(R_1)\mathrm{Det}(R_2) \neq 0$. R_1 is the incidence matrix of the subgraph of G induced by E_1 with the last column deleted. Since R_1 has full rank, this induced subgraph is a (spanning) tree. Likewise E_2 induces two (edge disjoint) trees. So we have a 3T2 decomposition which must be proper because we assumed G to be generically independent.

Given a proper 3T2 decomposition, we would like to produce an infinitesimally rigid embedding. We will start with a limit framework. If T_1, T_2 and T_3 are the trees of the 3T2 decomposition, then the sets of vertices of their pairwise intersections form a partition of V. Denote the vertices of $T_1 \cap T_2$ by $\bar{V}_3$, those of $T_2 \cap T_3$ by $\bar{V}_1$, and those of $T_3 \cap T_1$ by $\bar{V}_2$. At least two of these vertex sets are non-empty. Define a map $\mathbf{p}$ of the vertices by sending $\bar{V}_1$ to $(0,0)$, $\bar{V}_2$ to $(0,1)$, and $\bar{V}_3$ to $(1,0)$. It follows that all edges of T_1 which with distinct endpoints connect $(1,0)$ and $(0,1)$, and likewise for T_2 and T_3, see Figure 12.

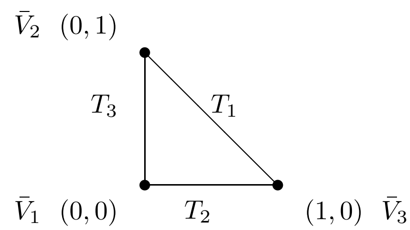


Figure 12: A limit framework for a 3T2 graph.

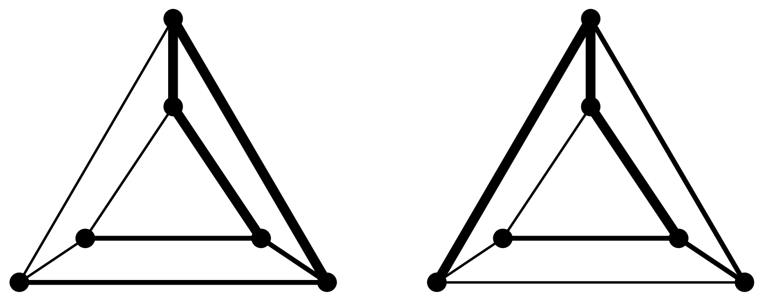
We want to show that $(V, E, \mathbf{p})$ is a limit framework such that all the edges of T_i, infinitesimal or not, are parallel. To do this we inductively peel apart the vertex sets $\bar{V}_1$, $\bar{V}_2$ and $\bar{V}_3$, one direction at a time, involving edges of one tree at a time. We can do this since, at any stage, the induced subgraph on those vertices mapped to some single point, say $\bar{V}_3$, consists of edges of just two of the trees, T_1 and T_2, and the intersection of that induced subgraph with one of the trees, say T_1, which join those vertices is disconnected, (otherwise there would be two subtrees with the same span.) We can then separate the connected components, in this case, in the horizontal direction.

Lastly we need to show that the rigidity matrix of the limit framework is of full rank. Reorder the rows of the rigidity matrix so the edges belonging to T_1 come first, then those of T_2, and then those of T_3. Reorder the columns in three sections by taking the columns corresponding to the first coordinates of $\bar{V}_2$ and the second coordinates of $\bar{V}_3$ in the first section, those corresponding to the first coordinates of $\bar{V}_1$ and $\bar{V}_3$ in the second section, and the second coordinates of $\bar{V}_1$ and $\bar{V}_2$ in the third.

$$\begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array} \left[\begin{array}{ccc} I_1 & ? & ? \\ 0 & I_2 & ? \\ 0 & 0 & I_3 \end{array} \right],$$

The matrix then has the block structure shown, with I_i being the incidence matrix of T_i, and so it has full rank. □

In the proof, each 3T2 decomposition gives a representation of the triangle as a limit framework. For the 3T2 decomposition of the triangular prism indicated in Figure 13a the limit framework is the one described in Figure 10. The 3T2



a. b.

Figure 13: 3T2 decompositions of the triangular prism.

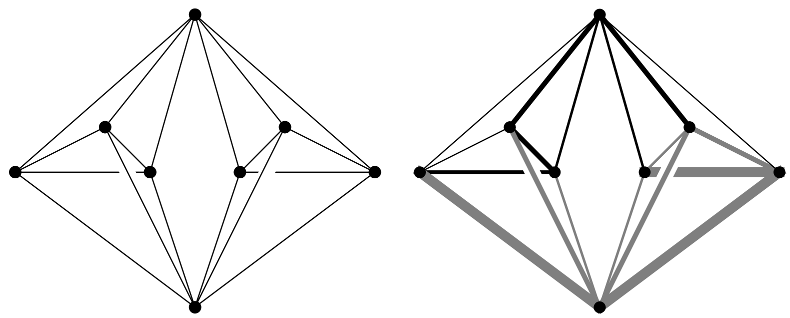
decomposition of the triangular prism of Figure 13b results in a limit framework that corresponds to the result of the vertical projection of the "screwing down" deformation of Figure 1c.

Tay's proof, [14, 15] can be generalized to hinged panel structures in m-space.

Generic Rigidity in 3-space

The most famous general result for generic rigidity in dimension three is that the obvious analogue of Laman's Theorem is not true. In dimension three a graph with n vertices needs at least $3n-6$ edges. We say that a graph G on n vertices satisfies *Laman's condition* if G has a subset F of $3n-6$ edges which spans all n vertices, and such that for any subset F' of F, $|F'| \leq 3|V(F')|-6$, where $V(E)$ denotes the set of all vertices which are endpoints of some edge in an edge set E. In other words, G has enough edges to be rigid but does not have any obviously overbraced subgraph. It is straightforward to show that every generically rigid graph in dimension 3 satisfies Laman's condition, however, the condition is not sufficient to insure generic rigidity.

Example: The "double banana" graph of Figure 14a is not generically rigid in dimension 3, but does satisfy Laman's condition. This graph is obviously flexible in any embedding since the two "bananas" can twist about the two vertex cutset. On the other hand it is easy to check that no subgraph is overbraced. □



a. b.

Figure 14: The double banana.

While Laman's condition does not characterize generic rigidity in 3 space, the other characterizations of planar rigidity are not known to fail. It may also be possible to augment Laman's condition. The obvious question, which remains unanswered, is whether every 3-connected graph satisfying Laman's condition is rigid.

The most promising approach is via tree decompositions. The three dimensional analogue to the 3T2 condition is to decompose the graph into 6 trees, each vertex belonging to exactly 3 of the trees. The remaining task here is to define "proper" correctly, since requiring merely that no two subtrees have the same span is not enough, since again the double banana is a counterexample, see Figure 14b.

For a complete discussion of the current state of affairs in 3D generic rigidity as well as possible approaches see [5].

Abstract Rigidity

It is possible to study infinitesimal and generic rigidity without reference to a specific rigidity matrix by looking only at how sets of edges depend on one another. Lovász and Yemini [8] were the first to explicitly state that matroid theory is the appropriate tool for studying generic rigidity. More recent treatments can be found in [5] and [17].

Recall that both infinitesimal and generic rigidity in dimension 1 are equivalent to the connectivity of the graph. Given a set E of edges of a complete graph, define the *closure* of that set, $\langle E \rangle$, as the set of all edges both of whose endpoints lie in the same connected component of the subgraph generated by E. It is easy to see that this closure operator is defined on the set of all subsets of the edges of a complete graph and satisfies

C1 $T \subseteq \langle T \rangle$;

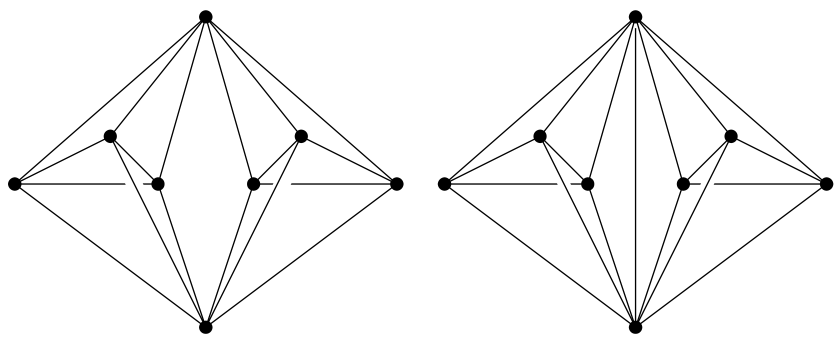
C2 If $R \subseteq T$, then $\langle R \rangle \subseteq \langle T \rangle$;

C3 $\langle \langle T \rangle \rangle = \langle T \rangle$.

C4 If $s, t \in (E - \langle T \rangle)$, then $s \in \langle T \cup \{t\} \rangle$ if and only if $t \in \langle T \cup \{s\} \rangle$.

Any set operator, defined on the set of all subsets of a fixed set S and which satisfies these conditions is said to form a *matroid* M on the set S. Given any matrix, there is a matroid defined on the set of its rows by setting the closure of any subset X of rows to be the set of all rows which are linear combinations of X. In this way, using the rigidity matrix, we can define a matroid on the set of edges of a framework in any dimension. In particular, for any embedding $\mathbf{p}$ of n vertices into m space, we can form a matroid $\mathcal{F}(\mathbf{p})$ on the edges of a complete graph called an *infinitesimal rigidity matroid*. If $\mathbf{p}$ is a generic embedding, then $\mathcal{G}_m(n) = \mathcal{F}(\mathbf{p})$ is called the *m-dimensional generic rigidity matroid on n vertices.*

$\mathcal{G}_1(n)$ is the *connectivity matroid* previously described. In general, for $(V, K, \mathbf{p})$ a framework on the complete graph, the closure of a set E of edges in $\mathcal{F}(\mathbf{p})$ consists of those edges which depend on E, that is, those pairs of vertices which are not infinitesimally expanded or contracted in any flex of $(V, E, \mathbf{p})$. In particular, the closure of any rigid set of edges is complete.



a. b.

Figure 15: The closure of the double banana.

Example: The closure of the set of edges of an infinitesimally rigid triangular prism in 2D is the set of edges of the complete graph on six vertices. If the prism has an infinitesimal flex, then its closure is itself. The closure of the double banana, Figure 15a, is the double banana together with the edge joining the two “hinge” vertices, Figure 15b. □

Recall that $V(E)$ denotes the set of all vertices which are endpoints of edges in E. For a set X of vertices, let $K(X)$ denote the edges of the complete graph on X. Observe that all infinitesimal and generic rigidity matroids satisfy the following extra conditions.

C5 If $E, F \subseteq K$ and $|V(E) \cap V(F)| < m$, then $\langle E \cup F \rangle \subseteq (K(V(E)) \cup K(V(F)))$.

C6 If $\langle E \rangle = K(V(E))$, $\langle F \rangle = K(V(F))$ and $|V(E) \cap V(F)| \geq m$, then $\langle E \cup F \rangle = K(V(E \cup F))$.

Briefly, the 5'th property comes from the fact that, if two subgraphs meet too few vertices, then there is an (infinitesimal) motion twisting the two graphs along their intersection, like in the case of the double banana. The 6'th property expresses that if two rigid subgraphs meet in sufficiently many vertices, then their union is rigid.

We call any matroid $\mathcal{A}_m$ on the edges of a complete graph which satisfies the two extra conditions an *m-dimensional abstract rigidity matroid.* A set of edges is said to be *rigid* in $\mathcal{A}_m$ if its closure is complete, $\langle E \rangle = K(E)$. Generally, a set of edges E is said to be *independent* if the closure of any proper subset of E is a proper subset of the closure of E, otherwise E is said to be *dependent.* A maximally independent set is called a *basis.* The *rank* of a set is the cardinality of its largest independent subset. A minimally dependent set is called a *cycle,* and a *cocycle* is a minimal set which intersects every basis.

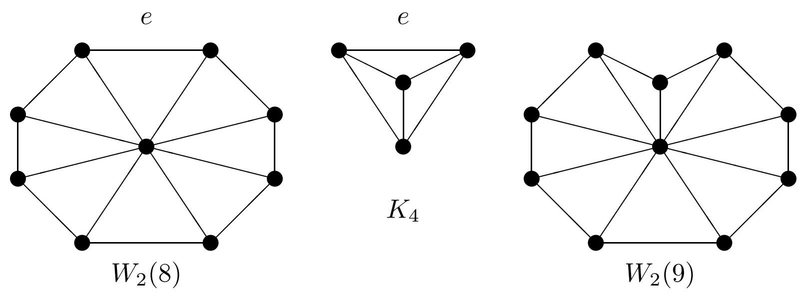


Figure 16: 2 dimensional wheels - cycles $\mathcal{A}_2$.

The connectivity matroid is the only abstract rigidity matroid in dimension 1, however there are many abstract rigidity matroids in higher dimensions. Nevertheless, they do contain many features in common. For example, it is not hard to show that every abstract rigidity matroid in dimension m has rank $m|V| - \binom{m+1}{2}$. In dimension 1, the cycles of the abstract rigidity matroid are the usual cycles in a graph. In dimensions greater than two, there are quite a variety of different cycles. In two dimensions, the edge sets of *wheels* are always cycles in $\mathcal{A}_2$, see Figure 16. The 3-dimensional wheels of Figure 17 must always be dependent in $\mathcal{A}_3$, although they need not be cycles.

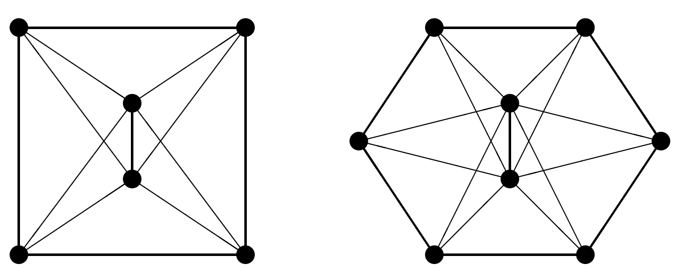


Figure 17: 3 dimensional wheels - dependent in $\mathcal{A}_3$.

Infinitesimal rigidity matroids have some obvious small cycles and cocycles. In the 1-dimensional generic rigidity matroid on the complete graph, i.e. the connectivity matroid, the triangle is the smallest cycle, and the star of any vertex is a cocycle. In dimension m, it is clear from elementary row operations that the set of edges of any complete graph on $m+2$ vertices, K_{m+2}, is a cycle, and also that the star of any vertex v minus any set A of $m-1$ edges incident to v, denoted by $S_A(v)$, is a cocycle, These facts are not only true for abstract rigidity matroids, they characterize them.

THEOREM 9 *A matroid $\mathcal{M}$ on the edge set of K_n is an m-dimensional abstract rigidity matroid if and only if all of the K_{m+2}'s are cycles and all of the $S_A(v)$'s are cocycles.*

From this theorem we have immediately that every infinitesimal rigidity matroid is an abstract rigidity matroid, however the converse is not true. We will close this section by constructing an abstract rigidity matroid which is not an infinitesimal rigidity matroid.

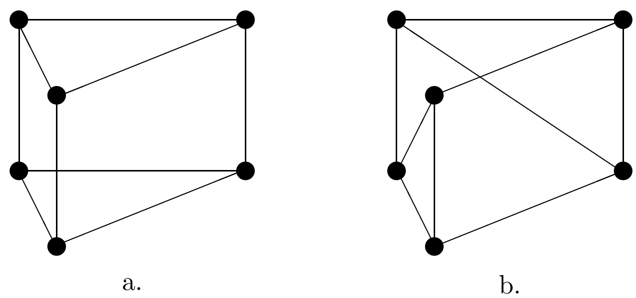


Figure 18: Prisms

The generic rigidity matroid on 6 vertices in the plane has rank $2 \cdot 6 - 3 = 9$, so every basis, that is, every a minimal rigid subgraph which spans the vertices, has 9 edges. Thus the edge set of any triangular prism is a basis in $\mathcal{G}_2(n)$ for $n \geq 6$. The set of edges of the triangular prism $(V, E, \mathbf{p})$ of Figure 18a is not a basis in $\mathcal{F}(\mathbf{p})$ since it is infinitesimally flexible, although the edge set of another prism on the same embedding may be a basis, Figure 18b. We want to construct an abstract rigidity matroid all of whose prisms are dependent.

We define a matroid on the edges of $K_6 = (V, E)$ as follows. Let $\mathcal{B}$ denote the collection of all subsets of E which are bases of $\mathcal{G}_2(6)$ with the exception of those which correspond to subgraphs isomorphic to the triangular prism. It is straightforward to verify that $\mathcal{B}$ is the collection of bases of a matroid, $\mathcal{M}$. To see that the matroid with bases $\mathcal{B}$ is an abstract rigidity matroid we note that the only difference between its closure operator and that of $\mathcal{G}_2(6)$ is that the closure of a prism, or a prism minus an edge is that prism, and no such edge set has either a separating vertex or is the union of two rigid subgraphs, so the axioms are satisfied.

To see that $\mathcal{M}$ is not an infinitesimal rigidity matroid, we have to show that there is no embedding $\mathbf{p}$ of K_6 into $\mathbb{R}^2$, so that the corresponding infinitesimal rigidity matroid is isomorphic to $\mathcal{M}$.

Any non-trivial infinitesimal motion of a prism, which we may assume to be zero on one of the triangles, must extend to an infinitesimal isometry of $\mathbb{R}^2$, which is either an infinitesimal translation or rotation, see Figure 19. If it is a

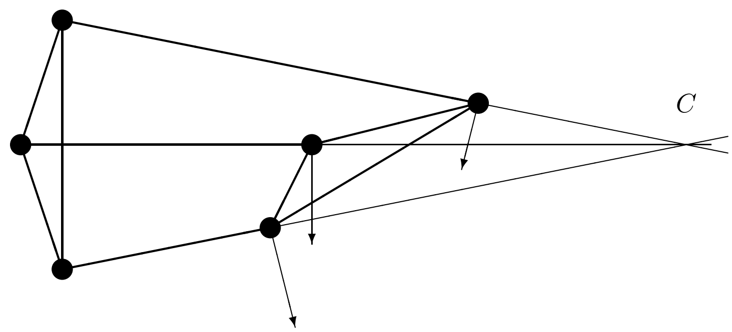


Figure 19: An infinitesimal motion on a prism

translation the lines connecting the triangles are parallel, while if it is a rotation, the lines must all pass through the center of rotation. Thus, in order for the edge set of a prism to be infinitesimally dependent, the three lines connecting the two triangles must be projectively concurrent. Since the 3 diagonal points of a complete quadrangle are never collinear in the projective plane, there is no general embedding of 6 points in $\mathbb{R}^2$ such that all prisms are dependent, and so $\mathcal{M} \neq \mathcal{F}(\mathbf{p})$ for any embedding $\mathbf{p}$.

If the edges E of a graph G are independent in some infinitesimal rigidity matroid $\mathcal{F}(\mathbf{p})$ in dimension m, then E must be independent in the generic rigidity matroid, as well as the edges of any subgraph isomorphic to G, and we write $\mathcal{F}(m) \leq \mathcal{G}_m(n)$. If the same were true for all abstract rigidity matroids, then this would yield a combinatorial characterization of $\mathcal{G}_m(n)$. This is known to be true in dimension less than three, and known to be false in dimension greater than 3.

CONJECTURE 1 (The Maximal Conjecture) *If $\mathcal{A}_3$ is any 3-dimensional abstract rigidity matroid on n vertices, then $\mathcal{G}_3(n) \geq \mathcal{A}_3$.*

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